

# A LICHNEROWICZ-HITCHIN VANISHING THEOREM FOR FOLIATIONS

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**ABSTRACT.** We establish a generalization of the Lichnerowicz-Hitchin vanishing theorem to the case of foliations. As a consequence, we show that there is no foliation of positive leafwise scalar curvature on any torus. Our proof, which is inspired by the analytic localization techniques developed by Bismut and Lebeau, also applies to give a new proof of the Connes vanishing theorem without using any noncommutative geometry.

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## 0. INTRODUCTION

A classical theorem of Lichnerowicz [15] states that if a closed spin manifold of dimension  $4k$  admits a Riemannian metric of positive scalar curvature, then the Hirzebruch

$\widehat{A}$ -genus<sup>1</sup> of this manifold vanishes. Hitchin [13] extended it to the case of all dimensions. In this paper, we generalize this Lichnerowicz-Hitchin result to the case of foliations.

To be more precise, let  $M$  be a smooth manifold, let  $F$  be an integrable subbundle of the tangent vector bundle  $TM$  of  $M$ . Let  $g^F$  be a Euclidean metric on  $F$ . Then  $g^F$  determines a leafwise scalar curvature  $k^F \in C^\infty(M)$  as follows: for any  $x \in M$ , the integrable subbundle  $F$  determines a leaf  $\mathcal{F}_x$  passing through  $x$  such that  $F|_{\mathcal{F}_x} = T\mathcal{F}_x$ . Thus,  $g^F$  determines a Riemannian metric on  $\mathcal{F}_x$ . Let  $k^{\mathcal{F}_x}$  denote the scalar curvature of this Riemannian metric. We define

$$(0.1) \quad k^F(x) = k^{\mathcal{F}_x}(x).$$

On the other hand, for a closed spin manifold  $M$ , let  $\widehat{\mathcal{A}}(M)$  be defined by that if  $\dim M = 8k + 4i$  with  $i = 0$  or  $1$ , then  $\widehat{\mathcal{A}}(M) = \frac{3+(-1)^{i+1}}{2}\widehat{A}(M)$ ; if  $\dim M = 8k + i$  with  $i = 1$  or  $2$ , then  $\widehat{\mathcal{A}}(M) \in \mathbf{Z}_2$  is the Atiyah-Milnor-Singer  $\alpha$  invariant<sup>2</sup>; while in other dimensions one takes  $\widehat{\mathcal{A}}(M) = 0$ .

The main result of this paper can be stated as follows.

**Theorem 0.1.** *Let  $F$  be an integrable subbundle of the tangent bundle of a closed spin manifold  $M$ . If there exists a metric  $g^F$  on  $F$  such that  $k^F > 0$  over  $M$ , then  $\widehat{\mathcal{A}}(M) = 0$ .*

When taking  $F = TM$ , one recovers the Lichnerowicz-Hitchin theorem.

Combining Theorem 0.1 with the well-known results of Gromov-Lawson [10] and Stolz [21], one gets the following purely geometric consequence.

**Corollary 0.2.** *Let  $F$  be an integrable subbundle of the tangent bundle of a closed simply connected manifold  $M$  with  $\dim M \geq 5$ . If there exists a metric  $g^F$  on  $F$  such that  $k^F > 0$  over  $M$ , then  $M$  admits a Riemannian metric of positive scalar curvature.*

**Remark 0.3.** That whether the existence of  $g^F$  with  $k^F > 0$  implies the existence of  $g^{TM}$  with  $k^{TM} > 0$  is a longstanding open question in foliation theory (cf. [24, Remark C14]), which admits an easy positive answer in the very special case where  $(M, F)$  carries a transverse Riemannian structure (cf. [8, page 8]). An approach to this question for codimension one foliations is outlined in [8, page 193].

Clearly, if the question in Remark 0.3 would have a positive answer, then Theorem 0.1 would be a direct consequence of the original Lichnerowicz-Hitchin theorem. The point here is that, conversely, while a direct geometric solution to this question is not available yet, the index theoretic results such as Theorem 0.1 can be used to study this purely geometric question.

On the other hand, as we will see, our proof of Theorem 0.1 applies to give a direct geometric proof of the following celebrated vanishing theorem of Connes, where instead of assuming  $TM$  being spin, one assumes that  $F$  is spin. This new proof provides a positive answer to a longstanding question in index theory (cf. [12, Page 5 of Lecture 9]).

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<sup>1</sup>Cf. [23, pp. 13] for a definition.

<sup>2</sup>Cf. [14, Section 2.7] for a definition.

**Theorem 0.4. (Connes [6])** *Let  $F$  be a spin integrable subbundle of the tangent bundle of a compact oriented manifold  $M$ . If there is a metric  $g^F$  on  $F$  such that  $k^F > 0$  over  $M$ , then  $\widehat{A}(M) = 0$ .*

Recall that the proof given in [6] for Theorem 0.4 is highly noncommutative. It is based on the Connes-Skandalis longitudinal index theorem for foliations [7] as well as the techniques of cyclic cohomology. Thus it relies essentially on the spin structure on  $F$ , and we do not see how to adapt this to prove Theorem 0.1.

However, we will make use of a geometric trick in [6], which is the construction of a fibration<sup>3</sup> over an arbitrary foliation, in our proof of Theorems 0.1 and 0.4. The key advantage of this fibration is that the lifted (from the original) foliation is almost isometric, i.e., very close to the Riemannian foliation on which we have seen that the question in Remark 0.3 admits an easy positive answer. On the other hand, this fibration is noncompact, which makes the proofs of both Theorems 0.1 and 0.4 highly nontrivial.

Roughly speaking, the Connes fibration over a foliation  $(M, F)$  is a fibration  $\pi : \mathcal{M} \rightarrow M$  where for any  $x \in M$ , the fiber  $\pi^{-1}(x)$  is the space of Euclidean metrics on the quotient space  $T_x M / F_x$ . The integrable subbundle  $F$  of  $TM$  lifts to an integrable subbundle  $\mathcal{F}$  of  $T\mathcal{M}$ , and  $(\mathcal{M}, \mathcal{F})$  carries an almost isometric structure in the sense of [6, Section 4]. Take any metric on the transverse bundle  $TM/F$ , which by definition determines an embedded section  $s : M \hookrightarrow \mathcal{M}$ . The induced fibration  $s \circ \pi : \mathcal{M} \rightarrow s(M)$  looks like a vector bundle, and Connes obtained his theorem by examining the corresponding Riemann-Roch property in noncommutative frameworks.

Our proof of Theorem 0.1 is different. It is inspired by the index theoretic analytic localization techniques developed by Bismut-Lebeau [4, Sections 8 and 9], and can be thought of as a kind of transgression. Moreover, it occurs as a surprise that for our proof, we need to work on a space larger than the Connes fibration itself.

To be more precise, let  $T^V \mathcal{M}$  be the vertical tangent bundle of the Connes fibration  $\pi : \mathcal{M} \rightarrow M$ . Then  $\mathcal{F} \oplus T^V \mathcal{M}$  is an integrable subbundle of  $T\mathcal{M}$ . Moreover, by definition  $T\mathcal{M}/(\mathcal{F} \oplus T^V \mathcal{M}) \simeq \pi^*(TM/F)$  carries a canonical metric. Thus, if  $\pi' : \widehat{\mathcal{M}} \rightarrow \mathcal{M}$  is the Connes fibration associated to the integrable subbundle  $\mathcal{F} \oplus T^V \mathcal{M}$  of  $T\mathcal{M}$ , then one obtains a canonical embedded section  $\widehat{s} : \mathcal{M} \hookrightarrow \widehat{\mathcal{M}}$ .

Now consider the fibration  $\widehat{\pi} = \pi' \circ \pi : \widehat{\mathcal{M}} \rightarrow M$  as well as the induced embedded section  $\widehat{s} = \widehat{s} \circ s : M \hookrightarrow \widehat{\mathcal{M}}$ . The integrable subbundle  $F$  of  $TM$  lifts to an integrable subbundle  $\widehat{\mathcal{F}}$  of  $T\widehat{\mathcal{M}}$ . Let  $T^V \widehat{\mathcal{M}}$  be the vertical tangent bundle of the fibration  $\widehat{\pi} : \widehat{\mathcal{M}} \rightarrow M$ , then it carries a natural metric  $g^{T^V \widehat{\mathcal{M}}}$ . Moreover, if we take a splitting

$$(0.2) \quad T\widehat{\mathcal{M}} = \widehat{\mathcal{F}} \oplus T^V \widehat{\mathcal{M}} \oplus \widehat{\mathcal{F}}^\perp,$$

then by definition  $\widehat{\mathcal{F}}^\perp \simeq (\pi')^*(T\mathcal{M}/(\mathcal{F} \oplus T^V \mathcal{M}))$  carries a natural metric  $g^{\widehat{\mathcal{F}}^\perp}$ .

Our first observation is that these metrics corresponding to the splitting (0.2) still provides  $(\widehat{\mathcal{M}}, \widehat{\mathcal{F}})$  with an almost isometric structure. If one lifts  $g^F$  to a metric  $g^{\widehat{\mathcal{F}}}$  on  $\widehat{\mathcal{F}}$ , then for any  $\beta > 0, \varepsilon > 0$ , one can consider the rescaled metric  $g_{\beta, \varepsilon}^{T\widehat{\mathcal{M}}} = \beta^2 g^{\widehat{\mathcal{F}}} \oplus g^{T^V \widehat{\mathcal{M}}} \oplus \frac{g^{\widehat{\mathcal{F}}^\perp}}{\varepsilon^2}$ .

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<sup>3</sup>Which will be called a Connes fibration in what follows.

Our second observation is that since  $TM$  is assumed to be spin,  $\widehat{\mathcal{F}} \oplus \widehat{\mathcal{F}}^\perp \simeq \widehat{\pi}^*(TM)$  is also spin. Thus one can construct a Dirac type operator<sup>4</sup>  $D_{\beta,\varepsilon}^{\widehat{\mathcal{M}}}$  acting on  $\Gamma(S(\widehat{\mathcal{F}} \oplus \widehat{\mathcal{F}}^\perp) \otimes \Lambda^*(T^V \widehat{\mathcal{M}}))$ , where  $S(\cdot)$  (resp.  $\Lambda^*(\cdot)$ ) is the notation for spinor bundle (resp. exterior algebra bundle).

Now take a sufficiently small open neighborhood  $U$  of  $\widetilde{s}(M)$  in  $\widehat{\mathcal{M}}$ . Inspired by [4], for any  $\beta, \varepsilon, T > 0$ , we construct an isometric embedding (see Section 2 for more details)

$$(0.3) \quad J_{T,\beta,\varepsilon} : \Gamma \left( S \left( \widehat{\mathcal{F}} \oplus \widehat{\mathcal{F}}^\perp \right) \Big|_{\widetilde{s}(M)} \right) \rightarrow \Gamma \left( S \left( \widehat{\mathcal{F}} \oplus \widehat{\mathcal{F}}^\perp \right) \otimes \Lambda^* \left( T^V \widehat{\mathcal{M}} \right) \right)$$

such that for any  $\sigma \in \Gamma(S(\widehat{\mathcal{F}} \oplus \widehat{\mathcal{F}}^\perp)|_{\widetilde{s}(M)})$ ,  $J_{T,\beta,\varepsilon}\sigma$  has compact support in  $U$ . Let  $E_{T,\beta,\varepsilon}$  be the  $L^2$ -completion of the image space of  $J_{T,\beta,\varepsilon}$ . Let  $p_{T,\beta,\varepsilon} : L^2(S(\widehat{\mathcal{F}} \oplus \widehat{\mathcal{F}}^\perp) \otimes \Lambda^*(T^V \widehat{\mathcal{M}})) \rightarrow E_{T,\beta,\varepsilon}$  be the orthogonal projection. Then one finds that the operator

$$(0.4) \quad J_{T,\beta,\varepsilon}^{-1} p_{T,\beta,\varepsilon} D_{\beta,\varepsilon}^{\widehat{\mathcal{M}}} J_{T,\beta,\varepsilon} : \Gamma \left( S \left( \widehat{\mathcal{F}} \oplus \widehat{\mathcal{F}}^\perp \right) \Big|_{\widetilde{s}(M)} \right) \rightarrow L^2 \left( S \left( \widehat{\mathcal{F}} \oplus \widehat{\mathcal{F}}^\perp \right) \Big|_{\widetilde{s}(M)} \right)$$

is elliptic, formally self-adjoint and homotopic to the Dirac operator on  $\widetilde{s}(M) \simeq M$ . Thus Theorem 0.1 will follow if one can show that for certain values of  $\beta, \varepsilon$  and  $T$ , this operator is invertible. Indeed, this is exactly what we will establish in this paper.

Moreover, by combining the above invertibility with the techniques of Lusztig [19] and Gromov-Lawson [9], one obtains the following result which generalizes the corresponding result of Schoen-Yau [20] and Gromov-Lawson [9] for the case of  $F = T(T^n)$ .

**Theorem 0.5.** *There exists no foliation  $(T^n, F)$  on any torus  $T^n$  such that the integrable subbundle  $F$  of  $T(T^n)$  carries a metric of positive scalar curvature over  $T^n$ .*

Now if we assume that  $F$  is spin instead of that  $TM$  being spin, we can replace the spinor bundle  $S(\widehat{\mathcal{F}} \oplus \widehat{\mathcal{F}}^\perp)$  in (0.3), (0.4) by  $S(\widehat{\mathcal{F}}) \otimes \Lambda^*(\widehat{\mathcal{F}}^\perp)$  and consider the corresponding sub-Dirac operators in the sense of [18]. In this way, we get a direct geometric proof of Theorem 0.4 without using any noncommutative geometry.

We would like to mention that the idea of constructing sub-Dirac operators has also been used in [16] to prove a generalization of the Atiyah-Hirzebruch vanishing theorem for circle actions [1] to the case of foliations.

This paper is organized as follows. In Section 1, we discuss the case of almost isometric foliations and carry out the local computation. We also introduce the sub-Dirac operator in this case and prove Theorem 0.4 in the case where the underlying foliation is compact. In Section 2, we work on noncompact Connes type fibrations and carry out the proofs of Theorems 0.1, 0.4 and 0.5.

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<sup>4</sup>Called a sub-Dirac operator in [18].

## 1. ADIABATIC LIMIT AND ALMOST ISOMETRIC FOLIATIONS

In this section, we discuss the geometry of almost isometric foliations in the sense of Connes [6]. We introduce for this kind of foliations a rescaled metric and show that the leafwise scalar curvature shows up from the limit behavior of the rescaled scalar curvature. We also introduce in this setting the sub-Dirac operators inspired by the original construction given in [18]. Finally, by combining the above two procedures, we prove a vanishing result when the almost isometric foliation under discussion is compact.

This section is organized as follows. In Section 1.1, we recall the definition of the almost isometric foliation in the sense of Connes. In Section 1.2 we introduce a rescaling of the given metric on the almost isometric foliation and study the corresponding limit behavior of the scalar curvature. In Section 1.3, we study Bott type connections on certain bundles transverse to the integrable subbundle. In Section 1.4, we introduce the so called sub-Dirac operator and compute the corresponding Lichnerowicz type formula. In Section 1.5 we prove a vanishing result when the almost isometric foliation is compact and verifies the conditions in Theorem 0.4.

**1.1. Almost isometric foliations.** Let  $(M, F)$  be a foliated manifold, where  $F$  is an integrable subbundle of  $TM$ , i.e., for any smooth sections  $X, Y \in \Gamma(F)$ , one has

$$(1.1) \quad [X, Y] \in \Gamma(F).$$

Let  $G$  be the holonomy groupoid of  $(M, F)$  (cf. [22]).

Let  $TM/F$  be the transverse bundle. We make the assumption that there is a proper subbundle  $E$  of  $TM/F$  and choose a splitting

$$(1.2) \quad TM/F = E \oplus (TM/F)/E.$$

Let  $q_1, q_2$  denote the dimensions of  $E$  and  $(TM/F)/E$  respectively.

**Definition 1.1. (Connes [6, Section 4])** If there exists a metric  $g^{TM/F}$  on  $TM/F$  with its restrictions to  $E$  and  $(TM/F)/E$  such that the action of  $G$  on  $TM/F$  takes the form

$$(1.3) \quad \begin{pmatrix} O(q_1) & 0 \\ A & O(q_2) \end{pmatrix},$$

where  $O(q_1), O(q_2)$  are orthogonal matrices of ranks  $q_1, q_2$  respectively, and  $A$  is a  $q_2 \times q_1$  matrix, then we say that  $(M, F)$  carries an almost isometric structure.

Clearly, the existence of the almost isometric structure does not depend on the splitting (1.2). We assume from now on that  $(M, F)$  carries an almost isometric structure as above.

Now choose a splitting  $TM = F \oplus F^\perp$ . We can and we will identify  $TM/F$  with  $F^\perp$ . Thus  $E$  and  $(TM/F)/E$  are identified with subbundles  $F_1^\perp, F_2^\perp$  of  $F^\perp$  respectively.

Let  $g^F$  be a metric on  $F$ . Let  $g^{F^\perp}$  be the metric on  $F^\perp$  corresponding to the metric  $g^{TM/F}$  and let  $g^{F_1^\perp}, g^{F_2^\perp}$  be the restrictions of  $g^{F^\perp}$  to  $F_1^\perp, F_2^\perp$ .

Let  $g^{TM}$  be a metric on  $TM$  so that we have the orthogonal splitting

$$(1.4) \quad TM = F \oplus F_1^\perp \oplus F_2^\perp, \quad g^{TM} = g^F \oplus g^{F_1^\perp} \oplus g^{F_2^\perp}.$$

Let  $\nabla^{TM}$  be the Levi-Civita connection associated to  $g^{TM}$ .

From the almost isometric condition (1.3), one deduces that for any  $X \in \Gamma(F)$ ,  $U_i, V_i \in \Gamma(F_i^\perp)$ ,  $i = 1, 2$ , the following identities, which may be thought of as infinitesimal versions of (1.3), hold (cf. [18, (A.5)]):

$$(1.5) \quad \begin{aligned} \langle [X, U_i], V_i \rangle + \langle U_i, [X, V_i] \rangle &= X \langle U_i, V_i \rangle, \\ \langle [X, U_2], U_1 \rangle &= 0. \end{aligned}$$

Equivalently,

$$(1.6) \quad \begin{aligned} \langle X, \nabla_{U_i}^{TM} V_i + \nabla_{V_i}^{TM} U_i \rangle &= 0, \\ \langle \nabla_X^{TM} U_2, U_1 \rangle + \langle X, \nabla_{U_2}^{TM} U_1 \rangle &= 0. \end{aligned}$$

In this paper, for simplicity, we also make the following assumption. This assumption holds by the Connes type fibration to be dealt with in the next section.

**Definition 1.2.** We call an almost isometric foliation as above verifies Condition (C) if  $F_2^\perp$  is also integrable. That is, for any  $U_2, V_2 \in \Gamma(F_2^\perp)$ , one has

$$(1.7) \quad [U_2, V_2] \in \Gamma(F_2^\perp).$$

**1.2. Adiabatic limit and the scalar curvature.** It has been shown in [18, Proposition A.2] that an almost isometric foliation in the sense of Definition 1.1 is an almost Riemannian foliation in the sense of [18, Definition 2.1]. Thus many computations in what follows are contained implicitly in [18] (see also [17]).

For convenience, we recall the standard formula for the Levi-Civita connection that for any  $X, Y, Z \in \Gamma(TM)$ ,

$$(1.8) \quad \begin{aligned} 2 \langle \nabla_X^{TM} Y, Z \rangle &= X \langle Y, Z \rangle + Y \langle X, Z \rangle - Z \langle X, Y \rangle \\ &\quad + \langle [X, Y], Z \rangle - \langle [X, Z], Y \rangle - \langle [Y, Z], X \rangle. \end{aligned}$$

For any  $\beta, \varepsilon > 0$ , let  $g_{\beta, \varepsilon}^{TM}$  be the rescaled Riemannian metric on  $TM$  defined by

$$(1.9) \quad g_{\beta, \varepsilon}^{TM} = \beta^2 g^F \oplus \frac{1}{\varepsilon^2} g^{F_1^\perp} \oplus g^{F_2^\perp}.$$

We will always assume that  $0 < \beta, \varepsilon \leq 1$ . We will use the subscripts and/or superscripts “ $\beta, \varepsilon$ ” to decorate the geometric data associated to  $g_{\beta, \varepsilon}^{TM}$ . For example,  $\nabla^{TM, \beta, \varepsilon}$  will denote the Levi-Civita connection associated to  $g_{\beta, \varepsilon}^{TM}$ . When the corresponding notation does not involve “ $\beta, \varepsilon$ ”, we will mean that it corresponds to the case of  $\beta = \varepsilon = 1$ .

Let  $p, p_1^\perp, p_2^\perp$  be the orthogonal projections from  $TM$  to  $F, F_1^\perp, F_2^\perp$  with respect to the orthogonal splitting (1.4). Let  $\nabla^{F, \beta, \varepsilon}, \nabla^{F_1^\perp, \beta, \varepsilon}, \nabla^{F_2^\perp, \beta, \varepsilon}$  be the Euclidean connections on  $F, F_1^\perp, F_2^\perp$  defined by

$$(1.10) \quad \nabla^{F, \beta, \varepsilon} = p \nabla^{TM, \beta, \varepsilon} p, \quad \nabla^{F_1^\perp, \beta, \varepsilon} = p_1^\perp \nabla^{TM, \beta, \varepsilon} p_1^\perp, \quad \nabla^{F_2^\perp, \beta, \varepsilon} = p_2^\perp \nabla^{TM, \beta, \varepsilon} p_2^\perp.$$

By (1.8)-(1.10) and the integrability of  $F$ , the following identities hold for  $X \in \Gamma(F)$ :

$$(1.11) \quad \begin{aligned} \nabla^{F, \beta, \varepsilon} &= \nabla^F := p \nabla^{TM} p, \quad p \nabla_X^{TM, \beta, \varepsilon} p_i^\perp = p \nabla_X^{TM} p_i^\perp, \quad i = 1, 2, \\ p_1^\perp \nabla_X^{TM, \beta, \varepsilon} p &= \beta^2 \varepsilon^2 p_1^\perp \nabla_X^{TM} p, \quad p_2^\perp \nabla_X^{TM, \beta, \varepsilon} p = \beta^2 p_2^\perp \nabla_X^{TM} p. \end{aligned}$$

From (1.5)-(1.9), we deduce that for  $X \in \Gamma(F)$ ,  $U_i, V_i \in \Gamma(F_i^\perp)$ ,  $i = 1, 2$ ,

$$(1.12) \quad \langle \nabla_{U_1}^{TM, \beta, \varepsilon} V_1, X \rangle = \langle \nabla_{U_1}^{TM} V_1, X \rangle = \frac{1}{2} \langle [U_1, V_1], X \rangle,$$

while

$$(1.13) \quad \langle \nabla_{U_2}^{TM, \beta, \varepsilon} V_2, X \rangle = \langle \nabla_{U_2}^{TM} V_2, X \rangle = \frac{1}{2} \langle [U_2, V_2], X \rangle = 0.$$

Equivalently, for any  $U_i \in \Gamma(F_i^\perp)$ ,  $i = 1, 2$ ,

$$(1.14) \quad p_1^\perp \nabla_{U_1}^{TM, \beta, \varepsilon} p = \beta^2 \varepsilon^2 p_1^\perp \nabla_{U_1}^{TM} p, \quad p_2^\perp \nabla_{U_2}^{TM, \beta, \varepsilon} p = 0.$$

Similarly, one verifies that

$$(1.15) \quad \begin{aligned} \langle \nabla_{U_1}^{TM, \beta, \varepsilon} X, U_2 \rangle &= \frac{1}{2} \langle [U_1, X], U_2 \rangle - \frac{\beta^2}{2} \langle [U_1, U_2], X \rangle, \\ \langle \nabla_{U_2}^{TM, \beta, \varepsilon} X, U_1 \rangle &= \frac{\varepsilon^2}{2} \langle [U_1, X], U_2 \rangle + \frac{\beta^2 \varepsilon^2}{2} \langle [U_1, U_2], X \rangle. \end{aligned}$$

For convenience of the later computations, we collect the asymptotic behavior of various covariant derivatives in the following lemma. These formulas can be derived by applying (1.5)-(1.9). The inner products appear in the lemma correspond to  $\beta = \varepsilon = 1$ .

**Lemma 1.3.** *The following formulas hold for  $X, Y, Z \in \Gamma(F)$ ,  $U_i, V_i, W_i \in \Gamma(F_i^\perp)$  with  $i = 1, 2$ , when  $\beta > 0$ ,  $\varepsilon > 0$  are small,*

$$(1.16) \quad \langle \nabla_X^{TM, \beta, \varepsilon} Y, Z \rangle = O(1), \quad \langle \nabla_X^{TM, \beta, \varepsilon} Y, U_1 \rangle = O(\beta^2 \varepsilon^2), \quad \langle \nabla_X^{TM, \beta, \varepsilon} Y, U_2 \rangle = O(\beta^2),$$

$$(1.17) \quad \langle \nabla_X^{TM, \beta, \varepsilon} U_1, Y \rangle = O(1), \quad \langle \nabla_X^{TM, \beta, \varepsilon} U_1, V_1 \rangle = O(1), \quad \langle \nabla_X^{TM, \beta, \varepsilon} U_1, U_2 \rangle = O(1),$$

$$(1.18) \quad \langle \nabla_X^{TM, \beta, \varepsilon} U_2, Y \rangle = O(1), \quad \langle \nabla_X^{TM, \beta, \varepsilon} U_2, U_1 \rangle = O(\varepsilon^2), \quad \langle \nabla_X^{TM, \beta, \varepsilon} U_2, V_2 \rangle = O(1),$$

$$(1.19) \quad \langle \nabla_{U_1}^{TM, \beta, \varepsilon} X, Y \rangle = O(1), \quad \langle \nabla_{U_1}^{TM, \beta, \varepsilon} X, V_1 \rangle = O(\beta^2 \varepsilon^2), \quad \langle \nabla_{U_1}^{TM, \beta, \varepsilon} X, U_2 \rangle = O(1),$$

$$(1.20) \quad \langle \nabla_{U_1}^{TM, \beta, \varepsilon} V_1, X \rangle = O(1), \quad \langle \nabla_{U_1}^{TM, \beta, \varepsilon} V_1, W_1 \rangle = O(1), \quad \langle \nabla_{U_1}^{TM, \beta, \varepsilon} V_1, U_2 \rangle = O\left(\frac{1}{\varepsilon^2}\right),$$

$$(1.21) \quad \langle \nabla_{U_1}^{TM, \beta, \varepsilon} U_2, X \rangle = O\left(\frac{1}{\beta^2}\right), \quad \langle \nabla_{U_1}^{TM, \beta, \varepsilon} U_2, V_1 \rangle = O(1), \quad \langle \nabla_{U_1}^{TM, \beta, \varepsilon} U_2, V_2 \rangle = O(1),$$

$$(1.22) \quad \langle \nabla_{U_2}^{TM, \beta, \varepsilon} X, Y \rangle = O(1), \quad \langle \nabla_{U_2}^{TM, \beta, \varepsilon} X, U_1 \rangle = O(\varepsilon^2), \quad \langle \nabla_{U_2}^{TM, \beta, \varepsilon} X, V_2 \rangle = 0,$$

(1.23)

$$\left\langle \nabla_{U_2}^{TM, \beta, \varepsilon} U_1, X \right\rangle = O\left(\frac{1}{\beta^2}\right), \quad \left\langle \nabla_{U_2}^{TM, \beta, \varepsilon} U_1, V_1 \right\rangle = O(1), \quad \left\langle \nabla_{U_2}^{TM, \beta, \varepsilon} U_1, V_2 \right\rangle = O(1),$$

$$(1.24) \quad \left\langle \nabla_{U_2}^{TM, \beta, \varepsilon} V_2, X \right\rangle = 0, \quad \left\langle \nabla_{U_2}^{TM, \beta, \varepsilon} V_2, U_1 \right\rangle = O(\varepsilon^2), \quad \left\langle \nabla_{U_2}^{TM, \beta, \varepsilon} V_2, W_2 \right\rangle = O(1).$$

In what follows, when we compute the asymptotics of various covariant derivatives, we will simply use the above asymptotic formulas freely without further notice.

Let  $R^{TM, \beta, \varepsilon} = (\nabla^{TM, \beta, \varepsilon})^2$  be the curvature of  $\nabla^{TM, \beta, \varepsilon}$ . Then for any  $X, Y \in \Gamma(TM)$ , one has the following standard formula,

$$(1.25) \quad R^{TM, \beta, \varepsilon}(X, Y) = \nabla_X^{TM, \beta, \varepsilon} \nabla_Y^{TM, \beta, \varepsilon} - \nabla_Y^{TM, \beta, \varepsilon} \nabla_X^{TM, \beta, \varepsilon} - \nabla_{[X, Y]}^{TM, \beta, \varepsilon}.$$

Let  $R^F = (\nabla^F)^2$  be the curvature of  $\nabla^F$ . Let  $k^{TM, \beta, \varepsilon}$ ,  $k^F$  denote the scalar curvature of  $g^{TM, \beta, \varepsilon}$ ,  $g^F$  respectively. Recall that  $k^F$  is defined in (0.1). The following formula for  $k^F$  is obvious,

$$(1.26) \quad k^F = - \sum_{i, j=1}^{\text{rk}(F)} \left\langle R^F(f_i, f_j) f_i, f_j \right\rangle,$$

where  $f_i$ ,  $i = 1, \dots, \text{rk}(F)$ , is an orthonormal basis of  $F$ . Clearly, when  $F = TM$ , it reduces to the usual definition of the scalar curvature  $k^{TM}$  of  $g^{TM}$ .

**Proposition 1.4.** *If Condition (C) holds, then when  $\beta > 0$ ,  $\varepsilon > 0$  are small, the following formula holds uniformly on any compact subset of  $M$ ,*

$$(1.27) \quad k^{TM, \beta, \varepsilon} = \frac{k^F}{\beta^2} + O\left(1 + \frac{\varepsilon^2}{\beta^2}\right).$$

*Proof.* By (1.1), (1.11), (1.25) and Lemma 1.3, one deduces that when  $\beta > 0$ ,  $\varepsilon > 0$  are very small, for any  $X, Y \in \Gamma(F)$ , one has

$$(1.28) \quad \begin{aligned} \left\langle R^{TM, \beta, \varepsilon}(X, Y) X, Y \right\rangle &= \left\langle \nabla_X^{TM, \beta, \varepsilon} (p + p_1^\perp + p_2^\perp) \nabla_Y^{TM, \beta, \varepsilon} X, Y \right\rangle \\ &\quad - \left\langle \nabla_Y^{TM, \beta, \varepsilon} (p + p_1^\perp + p_2^\perp) \nabla_X^{TM, \beta, \varepsilon} X, Y \right\rangle - \left\langle \nabla_{[X, Y]}^{TM, \beta, \varepsilon} X, Y \right\rangle \\ &= \left\langle R^F(X, Y) X, Y \right\rangle - \beta^2 \varepsilon^2 \left\langle p_1^\perp \nabla_Y^{TM} X, \nabla_X^{TM} Y \right\rangle - \beta^2 \left\langle p_2^\perp \nabla_Y^{TM} X, \nabla_X^{TM} Y \right\rangle \\ &\quad + \beta^2 \varepsilon^2 \left\langle p_1^\perp \nabla_X^{TM} X, \nabla_Y^{TM} Y \right\rangle + \beta^2 \left\langle p_2^\perp \nabla_X^{TM} X, \nabla_Y^{TM} Y \right\rangle \\ &= \left\langle R^F(X, Y) X, Y \right\rangle + O(\beta^2). \end{aligned}$$

For  $X \in \Gamma(F)$ ,  $U \in \Gamma(F_1^\perp)$ , by (1.5)-(1.25), one finds that when  $\beta, \varepsilon > 0$  are small,

$$(1.29) \quad \begin{aligned} \left\langle R^{TM, \beta, \varepsilon}(X, U) X, U \right\rangle &= \left\langle \nabla_X^{TM, \beta, \varepsilon} (p + p_1^\perp + p_2^\perp) \nabla_U^{TM, \beta, \varepsilon} X, U \right\rangle \\ &\quad - \left\langle \nabla_U^{TM, \beta, \varepsilon} (p + p_1^\perp + p_2^\perp) \nabla_X^{TM, \beta, \varepsilon} X, U \right\rangle - \left\langle \nabla_{(p+p_1^\perp+p_2^\perp)[X, U]}^{TM, \beta, \varepsilon} X, U \right\rangle \\ &= \beta^2 \varepsilon^2 \left\langle \nabla_X^{TM} p \nabla_U^{TM} X, U \right\rangle + \beta^2 \varepsilon^2 \left\langle \nabla_X^{TM, \beta, \varepsilon} p_1^\perp \nabla_U^{TM} X, U \right\rangle - \varepsilon^2 \left\langle p_2^\perp \nabla_U^{TM, \beta, \varepsilon} X, \nabla_X^{TM, \beta, \varepsilon} U \right\rangle \\ &\quad - \beta^2 \varepsilon^2 \left\langle \nabla_U^{TM} p \nabla_X^{TM} X, U \right\rangle - \beta^2 \varepsilon^2 \left\langle \nabla_U^{TM, \beta, \varepsilon} p_1^\perp \nabla_X^{TM} X, U \right\rangle + \varepsilon^2 \left\langle p_2^\perp \nabla_X^{TM, \beta, \varepsilon} X, \nabla_U^{TM, \beta, \varepsilon} U \right\rangle \end{aligned}$$



$$-\beta^2 \varepsilon^2 \left\langle \nabla_{(p+p_1^\perp)[X,U]}^{TM} X, U \right\rangle - \left\langle \nabla_{p_2^\perp[X,U]}^{TM,\beta,\varepsilon} X, U \right\rangle = O(\beta^2 + \varepsilon^2).$$

Similarly, for  $X \in \Gamma(F)$ ,  $U \in \Gamma(F_2^\perp)$ , one has that when  $\beta > 0$ ,  $\varepsilon > 0$  are small,

$$\begin{aligned} (1.30) \quad \langle R^{TM,\beta,\varepsilon}(X, U)X, U \rangle &= \left\langle \nabla_X^{TM,\beta,\varepsilon} (p + p_1^\perp + p_2^\perp) \nabla_U^{TM,\beta,\varepsilon} X, U \right\rangle \\ &\quad - \left\langle \nabla_U^{TM,\beta,\varepsilon} (p + p_1^\perp + p_2^\perp) \nabla_X^{TM,\beta,\varepsilon} X, U \right\rangle - \left\langle \nabla_{(p+p_1^\perp+p_2^\perp)[X,U]}^{TM,\beta,\varepsilon} X, U \right\rangle \\ &= \beta^2 \langle \nabla_X^{TM} p \nabla_U^{TM} X, U \rangle - \frac{1}{\varepsilon^2} \left\langle p_1^\perp \nabla_U^{TM,\beta,\varepsilon} X, \nabla_X^{TM,\beta,\varepsilon} U \right\rangle + \beta^2 \left\langle \nabla_X^{TM,\beta,\varepsilon} p_2^\perp \nabla_U^{TM} X, U \right\rangle \\ &\quad - \beta^2 \langle \nabla_U^{TM} p \nabla_X^{TM} X, U \rangle - \beta^2 \varepsilon^2 \left\langle \nabla_U^{TM,\beta,\varepsilon} p_1^\perp \nabla_X^{TM} X, U \right\rangle - \beta^2 \left\langle \nabla_U^{TM,\beta,\varepsilon} p_2^\perp \nabla_X^{TM} X, U \right\rangle \\ &\quad - \beta^2 \langle \nabla_{p[X,U]}^{TM} X, U \rangle - \beta^2 \left\langle \nabla_{p_2^\perp[X,U]}^{TM} X, U \right\rangle = O(\beta^2 + \varepsilon^2). \end{aligned}$$

For  $U, V \in \Gamma(F_1^\perp)$ , one verifies that

$$\begin{aligned} (1.31) \quad \langle R^{TM,\beta,\varepsilon}(U, V)U, V \rangle &= \left\langle \nabla_U^{TM,\beta,\varepsilon} (p + p_1^\perp + p_2^\perp) \nabla_V^{TM,\beta,\varepsilon} U, V \right\rangle \\ &\quad - \left\langle \nabla_V^{TM,\beta,\varepsilon} (p + p_1^\perp + p_2^\perp) \nabla_U^{TM,\beta,\varepsilon} U, V \right\rangle - \left\langle \nabla_{(p+p_1^\perp+p_2^\perp)[U,V]}^{TM,\beta,\varepsilon} U, V \right\rangle \\ &= \beta^2 \varepsilon^2 \left\langle \nabla_U^{TM} p \nabla_V^{TM,\beta,\varepsilon} U, V \right\rangle + \langle \nabla_U^{TM} p_1^\perp \nabla_V^{TM} U, V \rangle - \varepsilon^2 \left\langle p_2^\perp \nabla_V^{TM,\beta,\varepsilon} U, \nabla_U^{TM,\beta,\varepsilon} V \right\rangle \\ &\quad - \beta^2 \varepsilon^2 \left\langle \nabla_V^{TM} p \nabla_U^{TM,\beta,\varepsilon} U, V \right\rangle - \langle \nabla_V^{TM} p_1^\perp \nabla_U^{TM} U, V \rangle + \varepsilon^2 \left\langle p_2^\perp \nabla_U^{TM,\beta,\varepsilon} U, \nabla_V^{TM,\beta,\varepsilon} V \right\rangle \\ &\quad - \left\langle \nabla_{p[U,V]}^{TM,\beta,\varepsilon} U, V \right\rangle - \left\langle \nabla_{p_1^\perp[U,V]}^{TM} U, V \right\rangle - \left\langle \nabla_{p_2^\perp[U,V]}^{TM,\beta,\varepsilon} U, V \right\rangle \\ &= -\varepsilon^2 \left\langle p_2^\perp \nabla_V^{TM,\beta,\varepsilon} U, \nabla_U^{TM,\beta,\varepsilon} V \right\rangle + \varepsilon^2 \left\langle p_2^\perp \nabla_U^{TM,\beta,\varepsilon} U, \nabla_V^{TM,\beta,\varepsilon} V \right\rangle + O(1) = O\left(\frac{1}{\varepsilon^2}\right), \end{aligned}$$

from which one gets that when  $\beta > 0$ ,  $\varepsilon > 0$  are small,

$$(1.32) \quad \varepsilon^2 \langle R^{TM,\beta,\varepsilon}(U, V)U, V \rangle = O(1).$$

For  $U, V \in \Gamma(F_2^\perp)$ , one verifies directly that

$$\begin{aligned} (1.33) \quad \langle R^{TM,\beta,\varepsilon}(U, V)U, V \rangle &= \left\langle \nabla_U^{TM,\beta,\varepsilon} (p + p_1^\perp + p_2^\perp) \nabla_V^{TM,\beta,\varepsilon} U, V \right\rangle \\ &\quad - \left\langle \nabla_V^{TM,\beta,\varepsilon} (p + p_1^\perp + p_2^\perp) \nabla_U^{TM,\beta,\varepsilon} U, V \right\rangle - \left\langle \nabla_{[U,V]}^{TM,\beta,\varepsilon} U, V \right\rangle \\ &= \beta^2 \left\langle \nabla_U^{TM} p \nabla_V^{TM,\beta,\varepsilon} U, V \right\rangle - \frac{1}{\varepsilon^2} \left\langle p_1^\perp \nabla_V^{TM,\beta,\varepsilon} U, \nabla_U^{TM,\beta,\varepsilon} V \right\rangle + \langle \nabla_U^{TM} p_2^\perp \nabla_V^{TM} U, V \rangle \\ &\quad - \beta^2 \left\langle \nabla_V^{TM} p \nabla_U^{TM,\beta,\varepsilon} U, V \right\rangle + \frac{1}{\varepsilon^2} \left\langle p_1^\perp \nabla_U^{TM,\beta,\varepsilon} U, \nabla_V^{TM,\beta,\varepsilon} V \right\rangle - \langle \nabla_V^{TM} p_2^\perp \nabla_U^{TM} U, V \rangle \\ &\quad - \langle \nabla_{[U,V]}^{TM} U, V \rangle = O(1). \end{aligned}$$

For  $U \in \Gamma(F_1^\perp)$ ,  $V \in \Gamma(F_2^\perp)$ , one verifies directly that,

$$\begin{aligned}
(1.34) \quad \langle R^{TM,\beta,\varepsilon}(U, V)U, V \rangle &= \left\langle \nabla_U^{TM,\beta,\varepsilon} (p + p_1^\perp + p_2^\perp) \nabla_V^{TM,\beta,\varepsilon} U, V \right\rangle \\
&\quad - \left\langle \nabla_V^{TM,\beta,\varepsilon} (p + p_1^\perp + p_2^\perp) \nabla_U^{TM,\beta,\varepsilon} U, V \right\rangle - \left\langle \nabla_{[U,V]}^{TM,\beta,\varepsilon} U, V \right\rangle \\
&= -\beta^2 \left\langle p \nabla_V^{TM,\beta,\varepsilon} U, \nabla_U^{TM,\beta,\varepsilon} V \right\rangle - \frac{1}{\varepsilon^2} \left\langle p_1^\perp \nabla_V^{TM,\beta,\varepsilon} U, \nabla_U^{TM,\beta,\varepsilon} V \right\rangle + \left\langle \nabla_U^{TM,\beta,\varepsilon} p_2^\perp \nabla_V^{TM,\beta,\varepsilon} U, V \right\rangle \\
&\quad + \beta^2 \left\langle p \nabla_U^{TM,\beta,\varepsilon} U, \nabla_V^{TM,\beta,\varepsilon} V \right\rangle + \frac{1}{\varepsilon^2} \left\langle p_1^\perp \nabla_U^{TM,\beta,\varepsilon} U, \nabla_V^{TM,\beta,\varepsilon} V \right\rangle - \left\langle \nabla_V^{TM} p_2^\perp \nabla_U^{TM,\beta,\varepsilon} U, V \right\rangle \\
&\quad + \frac{1}{\varepsilon^2} \left\langle U, \nabla_{[U,V]}^{TM,\beta,\varepsilon} V \right\rangle = O\left(\frac{1}{\varepsilon^2} + \frac{1}{\beta^2}\right),
\end{aligned}$$

from which one gets that when  $\beta > 0$ ,  $\varepsilon > 0$  are small,

$$(1.35) \quad \varepsilon^2 \langle R^{TM,\beta,\varepsilon}(U, V)U, V \rangle = \langle R^{TM,\beta,\varepsilon}(V, U)V, U \rangle = O\left(1 + \frac{\varepsilon^2}{\beta^2}\right).$$

From (1.26), (1.28)-(1.30), (1.32), (1.33) and (1.35), one gets (1.27).  $\square$

**1.3. Bott connections on  $F_1^\perp$  and  $F_2^\perp$ .** From (1.5) and (1.7)-(1.10), one verifies directly that for  $X \in \Gamma(F)$ ,  $U_i, V_i \in \Gamma(F_i^\perp)$ ,  $i = 1, 2$ , one has

$$\begin{aligned}
(1.36) \quad \left\langle \nabla_X^{F_1^\perp, \beta, \varepsilon} U_1, V_1 \right\rangle &= \langle [X, U_1], V_1 \rangle - \frac{\beta^2 \varepsilon^2}{2} \langle [U_1, V_1], X \rangle, \\
\left\langle \nabla_X^{F_2^\perp, \beta, \varepsilon} U_2, V_2 \right\rangle &= \langle [X, U_2], V_2 \rangle.
\end{aligned}$$

By (1.36), one has that for  $X \in \Gamma(F)$ ,  $U_i \in \Gamma(F_i^\perp)$ ,  $i = 1, 2$ ,

$$(1.37) \quad \lim_{\varepsilon \rightarrow 0^+} \nabla_X^{F_i^\perp, \beta, \varepsilon} U_i = \widetilde{\nabla}_X^{F_i^\perp} U_i := p_i^\perp [X, U_i].$$

Let  $\widetilde{\nabla}^{F_i^\perp}$  be the connection on  $F_i^\perp$  defined by the second equality in (1.37) and by  $\widetilde{\nabla}_U^{F_i^\perp} U_i = \nabla_U^{F_i^\perp} U_i$  for  $U \in \Gamma(F^\perp) = \Gamma(F_1^\perp \oplus F_2^\perp)$ . In view of (1.37) and [5], we call  $\widetilde{\nabla}^{F_i^\perp}$  a Bott connection on  $F_i^\perp$  for  $i = 1$  or  $2$ . Let  $\widetilde{R}^{F_i^\perp}$  denote the curvature of  $\widetilde{\nabla}^{F_i^\perp}$  for  $i = 1, 2$ .

The following result holds without Condition (C).

**Lemma 1.5.** *For  $X, Y \in \Gamma(F)$  and  $i = 1, 2$ , the following identity holds,*

$$(1.38) \quad \widetilde{R}^{F_i^\perp}(X, Y) = 0.$$

*Proof.* We proceed as in [23, Proof of Lemma 1.14]. By (1.37) and the standard formula for the curvature (cf. [23, (1.3)]), for any  $U \in \Gamma(F_i^\perp)$ ,  $i = 1, 2$ , one has,

$$\begin{aligned}
(1.39) \quad \widetilde{R}^{F_i^\perp}(X, Y)U &= \widetilde{\nabla}_X^{F_i^\perp} \widetilde{\nabla}_Y^{F_i^\perp} U - \widetilde{\nabla}_Y^{F_i^\perp} \widetilde{\nabla}_X^{F_i^\perp} U - \widetilde{\nabla}_{[X,Y]}^{F_i^\perp} U \\
&= p_i^\perp ([X, [Y, U]] + [Y, [U, X]] + [U, [X, Y]]) - p_i^\perp [X, (\text{Id} - p_i^\perp) [Y, U]] \\
&\quad - p_i^\perp [Y, (\text{Id} - p_i^\perp) [U, X]] \\
&= -p_i^\perp [X, (p_1^\perp + p_2^\perp - p_i^\perp) [Y, U]] - p_i^\perp [Y, (p_1^\perp + p_2^\perp - p_i^\perp) [U, X]],
\end{aligned}$$

where the last equality follows from the Jacobi identity and the integrability of  $F$ .

Now if  $i = 1$ , then by (1.5), one has

$$(1.40) \quad p_1^\perp [X, p_2^\perp [Y, U]] = p_1^\perp [Y, p_2^\perp [U, X]] = 0.$$

While if  $i = 2$ , still by (1.5), one has

$$(1.41) \quad p_1^\perp [Y, U] = p_1^\perp [U, X] = 0.$$

From (1.39)-(1.41), one gets (1.38). The proof of Lemma 1.5 is completed.  $\square$

**Remark 1.6.** For  $i = 1, 2$ , let  $R^{F_i^\perp, \beta, \varepsilon}$  denote the curvature of  $\nabla^{F_i^\perp, \beta, \varepsilon}$ . From (1.36)-(1.38), one finds that for any  $X, Y \in \Gamma(F)$ , when  $\beta > 0$ ,  $\varepsilon > 0$  are small, the following identity holds:

$$(1.42) \quad R^{F_i^\perp, \beta, \varepsilon}(X, Y) = O(\beta^2 \varepsilon^2).$$

On the other hand, for  $i = 1, 2$ , and  $U_i, V_i, W_i, Z_i \in \Gamma(F_i^\perp)$ , by using (1.5), (1.7), (1.8), (1.10) and (1.25), one verifies directly that when  $\beta > 0$ ,  $\varepsilon > 0$  are small, the following identities, which will be used later, hold,

$$(1.43) \quad \beta^{-1} \varepsilon \left\langle R^{F_1^\perp, \beta, \varepsilon}(X, U_1) V_1, W_1 \right\rangle = O(\beta^{-1} \varepsilon),$$

$$(1.44) \quad \beta^{-1} \left\langle R^{F_2^\perp, \beta, \varepsilon}(X, U_2) V_2, W_2 \right\rangle = O(\beta^{-1}),$$

$$(1.45) \quad \beta^{-1} \left\langle R^{F_1^\perp, \beta, \varepsilon}(X, U_2) V_1, W_1 \right\rangle = O(\beta^{-1}),$$

$$(1.46) \quad \varepsilon^2 \left\langle R^{F_1^\perp, \beta, \varepsilon}(U_1, V_1) W_1, Z_1 \right\rangle = O(\varepsilon^2),$$

$$(1.47) \quad \left\langle R^{F_2^\perp, \beta, \varepsilon}(U_2, V_2) W_2, Z_2 \right\rangle = O(1),$$

$$(1.48) \quad \varepsilon \left\langle R^{F_1^\perp, \beta, \varepsilon}(U_1, U_2) V_1, W_1 \right\rangle = O(\varepsilon),$$

$$(1.49) \quad \left\langle R^{F_1^\perp, \beta, \varepsilon}(U_2, V_2) V_1, W_1 \right\rangle = O(1),$$

$$(1.50) \quad \beta^{-1} \varepsilon \left\langle R^{F_2^\perp, \beta, \varepsilon}(X, U_1) V_2, W_2 \right\rangle = O(\beta^{-1} \varepsilon),$$

$$(1.51) \quad \varepsilon \left\langle R^{F_2^\perp, \beta, \varepsilon}(U_1, U_2) V_2, W_2 \right\rangle = O(\varepsilon),$$

and

$$(1.52) \quad \varepsilon^2 \left\langle R^{F_2^\perp, \beta, \varepsilon}(U_1, V_1) V_2, W_2 \right\rangle = O(\varepsilon^2).$$

**1.4. Sub-Dirac operators associated to spin integrable subbundles.** Following [18, §2b], we assume now that  $TM$ ,  $F$ ,  $F_i^\perp$ ,  $i = 1, 2$ , are all oriented and of even rank, with the orientation of  $TM$  being compatible with the orientations on  $F$ ,  $F_1^\perp$  and  $F_2^\perp$  through (1.4). We further assume that  $F$  is spin and carries a fixed spin structure.

Let  $S(F) = S_+(F) \oplus S_-(F)$  be the Hermitian bundle of spinors associated to  $(F, g^F)$ . For any  $X \in \Gamma(F)$ , the Clifford action  $c(X)$  exchanges  $S_\pm(F)$ .

Let  $i = 1$  or  $2$ . Let  $\Lambda^*(F_i^\perp)$  denote the exterior algebra bundle of  $F_i^{\perp,*}$ . Then  $\Lambda^*(F_i^\perp)$  carries a canonically induced metric  $g^{\Lambda^*(F_i^\perp)}$  from  $g^{F_i^\perp}$ . For any  $U \in F_i^\perp$ , let  $U^* \in F_i^{\perp,*}$  correspond to  $U$  via  $g^{F_i^\perp}$ . For any  $U \in \Gamma(F_i^\perp)$ , set

$$(1.53) \quad c(U) = U^* \wedge -i_U, \quad \widehat{c}(U) = U^* \wedge +i_U,$$

where  $U^* \wedge$  and  $i_U$  are the exterior and interior multiplications by  $U^*$  and  $U$  on  $\Lambda^*(F_i^\perp)$ .

Denote  $q = \text{rk}(F)$ ,  $q_i = \text{rk}(F_i^\perp)$ .

Let  $h_1, \dots, h_{q_i}$  be an oriented orthonormal basis of  $F_i^\perp$ . Set

$$(1.54) \quad \tau(F_i^\perp, g^{F_i^\perp}) = \left( \frac{1}{\sqrt{-1}} \right)^{\frac{q_i(q_i+1)}{2}} c(h_1) \cdots c(h_{q_i}).$$

Then

$$(1.55) \quad \tau(F_i^\perp, g^{F_i^\perp})^2 = \text{Id}_{\Lambda^*(F_i^\perp)}.$$

Set

$$(1.56) \quad \Lambda_\pm^*(F_i^\perp) = \left\{ h \in \Lambda^*(F_i^\perp) : \tau(F_i^\perp, g^{F_i^\perp}) h = \pm h \right\}.$$

Since  $q_i$  is even, for any  $h \in F_i^\perp$ ,  $c(h)$  anti-commutes with  $\tau(F_i^\perp, g^{F_i^\perp})$ , while  $\widehat{c}(h)$  commutes with  $\tau(F_i^\perp, g^{F_i^\perp})$ . In particular,  $c(h)$  exchanges  $\Lambda_\pm^*(F_i^\perp)$ .

Let  $\widetilde{\tau}(F_i^\perp)$  denote the  $\mathbf{Z}_2$ -grading of  $\Lambda^*(F_i^\perp)$  defined by

$$(1.57) \quad \widetilde{\tau}(F_i^\perp) \big|_{\Lambda_{\text{odd}}^{\text{even}}(F_i^\perp)} = \pm \text{Id} \big|_{\Lambda_{\text{odd}}^{\text{even}}(F_i^\perp)}.$$

Now we have the following  $\mathbf{Z}_2$ -graded vector bundles over  $M$ :

$$(1.58) \quad S(F) = S_+(F) \oplus S_-(F),$$

$$(1.59) \quad \Lambda^*(F_i^\perp) = \Lambda_+^*(F_i^\perp) \oplus \Lambda_-^*(F_i^\perp), \quad i = 1, 2$$

and

$$(1.60) \quad \Lambda^*(F_i^\perp) = \Lambda^{\text{even}}(F_i^\perp) \oplus \Lambda^{\text{odd}}(F_i^\perp), \quad i = 1, 2.$$

We form the following  $\mathbf{Z}_2$ -graded tensor product, which will play a role in Section 2:

$$(1.61) \quad W(F, F_1^\perp, F_2^\perp) = S(F) \widehat{\otimes} \Lambda^*(F_1^\perp) \widehat{\otimes} \Lambda^*(F_2^\perp),$$

with the  $\mathbf{Z}_2$ -grading operator given by

$$(1.62) \quad \tau_W = \tau_{S(F)} \cdot \tau(F_1^\perp, g^{F_1^\perp}) \cdot \widetilde{\tau}(F_2^\perp),$$

where  $\tau_{S(F)}$  is the  $\mathbf{Z}_2$ -grading operator defining the splitting in (1.58). We denote by

$$(1.63) \quad W(F, F_1^\perp, F_2^\perp) = W_+(F, F_1^\perp, F_2^\perp) \oplus W_-(F, F_1^\perp, F_2^\perp)$$

the  $\mathbf{Z}_2$ -graded decomposition with respect to  $\tau_W$ .

Recall that the connections  $\nabla^F$ ,  $\nabla^{F_1^\perp}$  and  $\nabla^{F_2^\perp}$  have been defined in (1.10) with  $\beta = \varepsilon = 1$  there. They lift canonically to Hermitian connections  $\nabla^{S(F)}$ ,  $\nabla^{\Lambda^*(F_1^\perp)}$ ,  $\nabla^{\Lambda^*(F_2^\perp)}$  on  $S(F)$ ,  $\Lambda^*(F_1^\perp)$ ,  $\Lambda^*(F_2^\perp)$  respectively, preserving the corresponding  $\mathbf{Z}_2$ -gradings. Let  $\nabla^{W(F, F_1^\perp, F_2^\perp)}$  be the canonically induced connection on  $W(F, F_1^\perp, F_2^\perp)$  which preserves the canonically induced Hermitian metric on  $W(F, F_1^\perp, F_2^\perp)$ , and also the  $\mathbf{Z}_2$ -grading of  $W(F, F_1^\perp, F_2^\perp)$ .

For any vector bundle  $E$  over  $M$ , by an integral polynomial of  $E$  we will mean a bundle  $\phi(E)$  which is a polynomial in the exterior and symmetric powers of  $E$  with integral coefficients.

For  $i = 1, 2$ , let  $\phi_i(F_i^\perp)$  be an integral polynomial of  $F_i^\perp$ . We denote the complexification of  $\phi_i(F_i^\perp)$  by the same notation. Then  $\phi_i(F_i^\perp)$  carries a naturally induced Hermitian metric from  $g^{F_i^\perp}$  and also a naturally induced Hermitian connection  $\nabla^{\phi_i(F_i^\perp)}$  from  $\nabla^{F_i^\perp}$ .

Let  $W(F, F_1^\perp, F_2^\perp) \otimes \phi_1(F_1^\perp) \otimes \phi_2(F_2^\perp)$  be the  $\mathbf{Z}_2$ -graded vector bundle over  $M$ ,

$$(1.64) \quad W(F, F_1^\perp, F_2^\perp) \otimes \phi_1(F_1^\perp) \otimes \phi_2(F_2^\perp) = W_+(F, F_1^\perp, F_2^\perp) \otimes \phi_1(F_1^\perp) \otimes \phi_2(F_2^\perp) \\ \oplus W_-(F, F_1^\perp, F_2^\perp) \otimes \phi_1(F_1^\perp) \otimes \phi_2(F_2^\perp).$$

Let  $\nabla^{W \otimes \phi_1 \otimes \phi_2}$  denote the naturally induced Hermitian connection on  $W(F, F_1^\perp, F_2^\perp) \otimes \phi_1(F_1^\perp) \otimes \phi_2(F_2^\perp)$  with respect to the naturally induced Hermitian metric on it. Clearly,  $\nabla^{W \otimes \phi_1 \otimes \phi_2}$  preserves the  $\mathbf{Z}_2$ -graded decomposition in (1.64).

Let  $S$  be the  $\text{End}(TM)$ -valued one form on  $M$  defined by

$$(1.65) \quad \nabla^{TM} = \nabla^F + \nabla^{F_1^\perp} + \nabla^{F_2^\perp} + S.$$

Let  $e_1, \dots, e_{\dim M}$  be an orthonormal basis of  $TM$ . Let  $\nabla^{F, \phi_1(F_1^\perp) \otimes \phi_2(F_2^\perp)}$  be the Hermitian connection on  $W(F, F_1^\perp, F_2^\perp) \otimes \phi_1(F_1^\perp) \otimes \phi_2(F_2^\perp)$  defined by that for any  $X \in \Gamma(TM)$ ,

$$(1.66) \quad \nabla_X^{F, \phi_1(F_1^\perp) \otimes \phi_2(F_2^\perp)} = \nabla_X^{W \otimes \phi_1 \otimes \phi_2} + \frac{1}{4} \sum_{i,j=1}^{\dim M} \langle S(X) e_i, e_j \rangle c(e_i) c(e_j).$$

Let the linear operator  $D^{F, \phi_1(F_1^\perp) \otimes \phi_2(F_2^\perp)} : \Gamma(W(F, F_1^\perp, F_2^\perp) \otimes \phi_1(F_1^\perp) \otimes \phi_2(F_2^\perp)) \rightarrow \Gamma(W(F, F_1^\perp, F_2^\perp) \otimes \phi_1(F_1^\perp) \otimes \phi_2(F_2^\perp))$  be defined by (compare with [18, Definition 2.2])

$$(1.67) \quad D^{F, \phi_1(F_1^\perp) \otimes \phi_2(F_2^\perp)} = \sum_{i=1}^{\dim M} c(e_i) \nabla_{e_i}^{F, \phi_1(F_1^\perp) \otimes \phi_2(F_2^\perp)}.$$

We call  $D^{F, \phi_1(F_1^\perp) \otimes \phi_2(F_2^\perp)}$  a sub-Dirac operator with respect to the spin vector bundle  $F$ .

One verifies that  $D^{F, \phi_1(F_1^\perp) \otimes \phi_2(F_2^\perp)}$  is a first order formally self-adjoint elliptic differential operator. Moreover, it exchanges  $\Gamma(W_\pm(F, F_1^\perp, F_2^\perp) \otimes \phi_1(F_1^\perp) \otimes \phi_2(F_2^\perp))$ . We denote by  $D_\pm^{F, \phi_1(F_1^\perp) \otimes \phi_2(F_2^\perp)}$  the restrictions of  $D^{F, \phi_1(F_1^\perp) \otimes \phi_2(F_2^\perp)}$  to  $\Gamma(W_\pm(F, F_1^\perp, F_2^\perp) \otimes \phi_1(F_1^\perp) \otimes \phi_2(F_2^\perp))$ . Then one has

$$(1.68) \quad \left( D_+^{F, \phi_1(F_1^\perp) \otimes \phi_2(F_2^\perp)} \right)^* = D_-^{F, \phi_1(F_1^\perp) \otimes \phi_2(F_2^\perp)}.$$

**Remark 1.7.** As in [18, (2.21)], when  $F_1^\perp, F_2^\perp$  are also spin and carry fixed spin structures, then  $TM = F \oplus F_1^\perp \oplus F_2^\perp$  is spin and carries an induced spin structure from the

spin structures on  $F$ ,  $F_1^\perp$  and  $F_2^\perp$ . Moreover, one has the following identifications of  $\mathbf{Z}_2$ -graded vector bundles (cf. [14]) for  $i = 1, 2$ ,

$$(1.69) \quad \Lambda_+^*(F_i^\perp) \oplus \Lambda_-^*(F_i^\perp) = S_+(F_i^\perp) \otimes S(F_i^\perp)^* \oplus S_-(F_i^\perp) \otimes S(F_i^\perp)^*,$$

$$(1.70) \quad \Lambda^{\text{even}}(F_i^\perp) \oplus \Lambda^{\text{odd}}(F_i^\perp) = \left( S_+(F_i^\perp) \otimes S_+(F_i^\perp)^* \oplus S_-(F_i^\perp) \otimes S_-(F_i^\perp)^* \right) \\ \oplus \left( S_+(F_i^\perp) \otimes S_-(F_i^\perp)^* \oplus S_-(F_i^\perp) \otimes S_+(F_i^\perp)^* \right).$$

By (1.54)-(1.67), (1.69) and (1.70),  $D_{F, \phi_1(F_1^\perp) \otimes \phi_2(F_2^\perp)}$  is simply the twisted Dirac operator

$$(1.71) \quad D^{F, \phi_1(F_1^\perp) \otimes \phi_2(F_2^\perp)} : \Gamma \left( S(TM) \widehat{\otimes} S(F_2^\perp)^* \otimes S(F_1^\perp)^* \otimes \phi_1(F_1^\perp) \otimes \phi_2(F_2^\perp) \right) \\ \longrightarrow \Gamma \left( S(TM) \widehat{\otimes} S(F_2^\perp)^* \otimes S(F_1^\perp)^* \otimes \phi_1(F_1^\perp) \otimes \phi_2(F_2^\perp) \right),$$

where for  $i = 1, 2$ , the Hermitian (dual) bundle of spinors  $S(F_i^\perp)^*$  associated to  $(F_i^\perp, g^{F_i^\perp})$  carries the Hermitian connection induced from  $\nabla^{F_i^\perp}$ .

The point of (1.67) is that it only requires  $F$  being spin. While on the other hand, (1.71) allows us to take the advantage of applying the calculations already done for usual (twisted) Dirac operators when doing local computations.

**Remark 1.8.** It is clear that the definition in (1.67) does not require that  $F$  being an integrable subbundle of  $TM$ . It applies to any splitting of  $TM$  in (1.4).

Let  $\Delta^{F, \phi_1(F_1^\perp) \otimes \phi_2(F_2^\perp)}$  denote the Bochner Laplacian defined by

$$(1.72) \quad \Delta^{F, \phi_1(F_1^\perp) \otimes \phi_2(F_2^\perp)} = \sum_{i=1}^{\dim M} \left( \nabla_{e_i}^{F, \phi_1(F_1^\perp) \otimes \phi_2(F_2^\perp)} \right)^2 - \nabla_{\sum_{i=1}^{\dim M} \nabla_{e_i}^{TM} e_i}^{F, \phi_1(F_1^\perp) \otimes \phi_2(F_2^\perp)}.$$

Let  $f_1, \dots, f_q$  be an oriented orthonormal basis of  $F$ . Let  $h_1, \dots, h_{q_1}$  (resp.  $e_1, \dots, e_{q_2}$ ) be an oriented orthonormal basis of  $F_1^\perp$  (resp.  $F_2^\perp$ ).

Let  $k^{TM}$  be the scalar curvature of  $g^{TM}$ ,  $R^{F_i^\perp}$  ( $i = 1, 2$ ) be the curvature of  $\nabla^{F_i^\perp}$ . Let  $R^{\phi_1(F_1^\perp) \otimes \phi_2(F_2^\perp)}$  be the curvature of the tensor product connection on  $\phi_1(F_1^\perp) \otimes \phi_2(F_2^\perp)$  induced from  $\nabla^{\phi_1(F_1^\perp)}$  and  $\nabla^{\phi_2(F_2^\perp)}$ .

In view of Remark 1.7, the following Lichnerowicz type formula, which is an analogue of [18, Theorem 2.3], holds:

$$(1.73) \quad \left( D^{F, \phi_1(F_1^\perp) \otimes \phi_2(F_2^\perp)} \right)^2 = -\Delta^{F, \phi_1(F_1^\perp) \otimes \phi_2(F_2^\perp)} + \frac{k^{TM}}{4} \\ + \frac{1}{2} \sum_{i,j=1}^q c(f_i) c(f_j) R^{\phi_1(F_1^\perp) \otimes \phi_2(F_2^\perp)}(f_i, f_j) + \frac{1}{2} \sum_{i,j=1}^{q_1} c(h_i) c(h_j) R^{\phi_1(F_1^\perp) \otimes \phi_2(F_2^\perp)}(h_i, h_j) \\ + \frac{1}{2} \sum_{i,j=1}^{q_2} c(e_i) c(e_j) R^{\phi_1(F_1^\perp) \otimes \phi_2(F_2^\perp)}(e_i, e_j) + \sum_{i=1}^q \sum_{j=1}^{q_1} c(f_i) c(h_j) R^{\phi_1(F_1^\perp) \otimes \phi_2(F_2^\perp)}(f_i, h_j) \\ + \sum_{i=1}^q \sum_{j=1}^{q_2} c(f_i) c(e_j) R^{\phi_1(F_1^\perp) \otimes \phi_2(F_2^\perp)}(f_i, e_j) + \sum_{i=1}^{q_1} \sum_{j=1}^{q_2} c(h_i) c(e_j) R^{\phi_1(F_1^\perp) \otimes \phi_2(F_2^\perp)}(h_i, e_j)$$

$$\begin{aligned}
& + \frac{1}{8} \sum_{i,j=1}^q \sum_{s,t=1}^{q_1} \left\langle R^{F_1^\perp} (f_i, f_j) h_t, h_s \right\rangle c(f_i) c(f_j) \widehat{c}(h_s) \widehat{c}(h_t) \\
& + \frac{1}{8} \sum_{i,j=1}^{q_1} \sum_{s,t=1}^{q_1} \left\langle R^{F_1^\perp} (h_i, h_j) h_t, h_s \right\rangle c(h_i) c(h_j) \widehat{c}(h_s) \widehat{c}(h_t) \\
& + \frac{1}{8} \sum_{i,j=1}^{q_2} \sum_{s,t=1}^{q_1} \left\langle R^{F_1^\perp} (e_i, e_j) h_t, h_s \right\rangle c(e_i) c(e_j) \widehat{c}(h_s) \widehat{c}(h_t) \\
& + \frac{1}{4} \sum_{i=1}^q \sum_{j=1}^{q_1} \sum_{s,t=1}^{q_1} \left\langle R^{F_1^\perp} (f_i, h_j) h_t, h_s \right\rangle c(f_i) c(h_j) \widehat{c}(h_s) \widehat{c}(h_t) \\
& + \frac{1}{4} \sum_{i=1}^q \sum_{j=1}^{q_2} \sum_{s,t=1}^{q_1} \left\langle R^{F_1^\perp} (f_i, e_j) h_t, h_s \right\rangle c(f_i) c(e_j) \widehat{c}(h_s) \widehat{c}(h_t) \\
& + \frac{1}{4} \sum_{i=1}^{q_1} \sum_{j=1}^{q_2} \sum_{s,t=1}^{q_1} \left\langle R^{F_1^\perp} (h_i, e_j) h_t, h_s \right\rangle c(h_i) c(e_j) \widehat{c}(h_s) \widehat{c}(h_t) \\
& + \frac{1}{8} \sum_{i,j=1}^q \sum_{s,t=1}^{q_2} \left\langle R^{F_2^\perp} (f_i, f_j) e_t, e_s \right\rangle c(f_i) c(f_j) \widehat{c}(e_s) \widehat{c}(e_t) \\
& + \frac{1}{8} \sum_{i,j=1}^{q_1} \sum_{s,t=1}^{q_2} \left\langle R^{F_2^\perp} (h_i, h_j) e_t, e_s \right\rangle c(h_i) c(h_j) \widehat{c}(e_s) \widehat{c}(e_t) \\
& + \frac{1}{8} \sum_{i,j=1}^{q_2} \sum_{s,t=1}^{q_2} \left\langle R^{F_2^\perp} (e_i, e_j) e_t, e_s \right\rangle c(e_i) c(e_j) \widehat{c}(e_s) \widehat{c}(e_t) \\
& + \frac{1}{4} \sum_{i=1}^q \sum_{j=1}^{q_1} \sum_{s,t=1}^{q_2} \left\langle R^{F_2^\perp} (f_i, h_j) e_t, e_s \right\rangle c(f_i) c(h_j) \widehat{c}(e_s) \widehat{c}(e_t) \\
& + \frac{1}{4} \sum_{i=1}^q \sum_{j=1}^{q_2} \sum_{s,t=1}^{q_2} \left\langle R^{F_2^\perp} (f_i, e_j) e_t, e_s \right\rangle c(f_i) c(e_j) \widehat{c}(e_s) \widehat{c}(e_t) \\
& + \frac{1}{4} \sum_{i=1}^{q_1} \sum_{j=1}^{q_2} \sum_{s,t=1}^{q_2} \left\langle R^{F_2^\perp} (h_i, e_j) e_t, e_s \right\rangle c(h_i) c(e_j) \widehat{c}(e_s) \widehat{c}(e_t).
\end{aligned}$$

When  $M$  is compact, by the Atiyah-Singer index theorem [2] (cf. [14]), one has

$$\begin{aligned}
(1.74) \quad \text{ind} \left( D_+^{F, \phi_1(F_1^\perp) \otimes \phi_2(F_2^\perp)} \right) \\
= 2^{\frac{q_1}{2}} \left\langle \widehat{A}(F) \widehat{L}(F_1^\perp) e(F_2^\perp) \text{ch}(\phi_1(F_1^\perp)) \text{ch}(\phi_2(F_2^\perp)), [M] \right\rangle,
\end{aligned}$$

where  $\widehat{L}(F_1^\perp)$  is the Hirzebruch  $\widehat{L}$ -class (cf. [14, (11.18') of Chap. III]) of  $F_1^\perp$ ,  $e(F_2^\perp)$  is the Euler class (cf. [23, §3.4]) of  $F_2^\perp$ , and “ch” is the notation for the Chern character (cf. [23, §1.6.4]).

**1.5. A vanishing theorem for almost isometric foliations.** In this subsection, we assume  $M$  is compact and prove a vanishing theorem for it. Some of the computations in this subsection will be used in the next section where we will deal with the case where  $M$  is non-compact.

Let  $\beta > 0$ ,  $\varepsilon > 0$  and consider the construction in Section 1.4 with respect to the metric  $g_{\beta,\varepsilon}^{TM}$  defined in (1.9). We still use the superscripts “ $\beta, \varepsilon$ ” to decorate the geometric data associated to  $g_{\beta,\varepsilon}^{TM}$ . For example,  $D^{F,\phi_1(F_1^\perp)\otimes\phi_2(F_2^\perp),\beta,\varepsilon}$  now denotes the sub-Dirac operator constructed in (1.67) associated to  $g_{\beta,\varepsilon}^{TM}$ . Moreover, it can be written as

(1.75)

$$\begin{aligned} D^{F,\phi_1(F_1^\perp)\otimes\phi_2(F_2^\perp),\beta,\varepsilon} = & \beta^{-1} \sum_{i=1}^q c(f_i) \nabla_{f_i}^{F,\phi_1(F_1^\perp)\otimes\phi_2(F_2^\perp),\beta,\varepsilon} + \varepsilon \sum_{j=1}^{q_1} c(h_j) \nabla_{h_j}^{F,\phi_1(F_1^\perp)\otimes\phi_2(F_2^\perp),\beta,\varepsilon} \\ & + \sum_{s=1}^{q_2} c(e_s) \nabla_{e_s}^{F,\phi_1(F_1^\perp)\otimes\phi_2(F_2^\perp),\beta,\varepsilon}. \end{aligned}$$

By (1.75), the Lichnerowicz type formula (1.73) for  $(D^{F,\phi_1(F_1^\perp)\otimes\phi_2(F_2^\perp),\beta,\varepsilon})^2$  takes the following form,

$$\begin{aligned} (1.76) \quad & \left( D^{F,\phi_1(F_1^\perp)\otimes\phi_2(F_2^\perp),\beta,\varepsilon} \right)^2 = -\Delta^{F,\phi_1(F_1^\perp)\otimes\phi_2(F_2^\perp),\beta,\varepsilon} + \frac{k^{TM,\beta,\varepsilon}}{4} \\ & + \frac{1}{2\beta^2} \sum_{i,j=1}^q c(f_i) c(f_j) R^{\phi_1(F_1^\perp)\otimes\phi_2(F_2^\perp),\beta,\varepsilon}(f_i, f_j) + \frac{\varepsilon^2}{2} \sum_{i,j=1}^{q_1} c(h_i) c(h_j) R^{\phi_1(F_1^\perp)\otimes\phi_2(F_2^\perp),\beta,\varepsilon}(h_i, h_j) \\ & + \frac{1}{2} \sum_{i,j=1}^{q_2} c(e_i) c(e_j) R^{\phi_1(F_1^\perp)\otimes\phi_2(F_2^\perp),\beta,\varepsilon}(e_i, e_j) + \frac{\varepsilon}{\beta} \sum_{i=1}^q \sum_{j=1}^{q_1} c(f_i) c(h_j) R^{\phi_1(F_1^\perp)\otimes\phi_2(F_2^\perp),\beta,\varepsilon}(f_i, h_j) \\ & + \frac{1}{\beta} \sum_{i=1}^q \sum_{j=1}^{q_2} c(f_i) c(e_j) R^{\phi_1(F_1^\perp)\otimes\phi_2(F_2^\perp),\beta,\varepsilon}(f_i, e_j) + \varepsilon \sum_{i=1}^{q_1} \sum_{j=1}^{q_2} c(h_i) c(e_j) R^{\phi_1(F_1^\perp)\otimes\phi_2(F_2^\perp),\beta,\varepsilon}(h_i, e_j) \\ & + \frac{1}{8\beta^2} \sum_{i,j=1}^q \sum_{s,t=1}^{q_1} \left\langle R^{F_1^\perp,\beta,\varepsilon}(f_i, f_j) h_t, h_s \right\rangle c(f_i) c(f_j) \widehat{c}(h_s) \widehat{c}(h_t) \\ & + \frac{\varepsilon^2}{8} \sum_{i,j=1}^{q_1} \sum_{s,t=1}^{q_1} \left\langle R^{F_1^\perp,\beta,\varepsilon}(h_i, h_j) h_t, h_s \right\rangle c(h_i) c(h_j) \widehat{c}(h_s) \widehat{c}(h_t) \\ & + \frac{1}{8} \sum_{i,j=1}^{q_2} \sum_{s,t=1}^{q_1} \left\langle R^{F_1^\perp,\beta,\varepsilon}(e_i, e_j) h_t, h_s \right\rangle c(e_i) c(e_j) \widehat{c}(h_s) \widehat{c}(h_t) \\ & + \frac{\varepsilon}{4\beta} \sum_{i=1}^q \sum_{j=1}^{q_1} \sum_{s,t=1}^{q_1} \left\langle R^{F_1^\perp,\beta,\varepsilon}(f_i, h_j) h_t, h_s \right\rangle c(f_i) c(h_j) \widehat{c}(h_s) \widehat{c}(h_t) \\ & + \frac{1}{4\beta} \sum_{i=1}^q \sum_{j=1}^{q_2} \sum_{s,t=1}^{q_1} \left\langle R^{F_1^\perp,\beta,\varepsilon}(f_i, e_j) h_t, h_s \right\rangle c(f_i) c(e_j) \widehat{c}(h_s) \widehat{c}(h_t) \\ & + \frac{\varepsilon}{4} \sum_{i=1}^{q_1} \sum_{j=1}^{q_2} \sum_{s,t=1}^{q_1} \left\langle R^{F_1^\perp,\beta,\varepsilon}(h_i, e_j) h_t, h_s \right\rangle c(h_i) c(e_j) \widehat{c}(h_s) \widehat{c}(h_t) \end{aligned}$$



$$\begin{aligned}
& + \frac{1}{8\beta^2} \sum_{i,j=1}^q \sum_{s,t=1}^{q_2} \left\langle R^{F_2^\perp, \beta, \varepsilon} (f_i, f_j) e_t, e_s \right\rangle c(f_i) c(f_j) \widehat{c}(e_s) \widehat{c}(e_t) \\
& + \frac{\varepsilon^2}{8} \sum_{i,j=1}^{q_1} \sum_{s,t=1}^{q_2} \left\langle R^{F_2^\perp, \beta, \varepsilon} (h_i, h_j) e_t, e_s \right\rangle c(h_i) c(h_j) \widehat{c}(e_s) \widehat{c}(e_t) \\
& + \frac{1}{8} \sum_{i,j=1}^{q_2} \sum_{s,t=1}^{q_2} \left\langle R^{F_2^\perp, \beta, \varepsilon} (e_i, e_j) e_t, e_s \right\rangle c(e_i) c(e_j) \widehat{c}(e_s) \widehat{c}(e_t) \\
& + \frac{\varepsilon}{4\beta} \sum_{i=1}^q \sum_{j=1}^{q_1} \sum_{s,t=1}^{q_2} \left\langle R^{F_2^\perp, \beta, \varepsilon} (f_i, h_j) e_t, e_s \right\rangle c(f_i) c(h_j) \widehat{c}(e_s) \widehat{c}(e_t) \\
& + \frac{1}{4\beta} \sum_{i=1}^q \sum_{j=1}^{q_2} \sum_{s,t=1}^{q_2} \left\langle R^{F_2^\perp, \beta, \varepsilon} (f_i, e_j) e_t, e_s \right\rangle c(f_i) c(e_j) \widehat{c}(e_s) \widehat{c}(e_t) \\
& + \frac{\varepsilon}{4} \sum_{i=1}^{q_1} \sum_{j=1}^{q_2} \sum_{s,t=1}^{q_2} \left\langle R^{F_2^\perp, \beta, \varepsilon} (h_i, e_j) e_t, e_s \right\rangle c(h_i) c(e_j) \widehat{c}(e_s) \widehat{c}(e_t).
\end{aligned}$$

By (1.27), (1.42)-(1.52) and (1.76), we get that when  $\beta > 0$ ,  $\varepsilon > 0$  are small,

$$(1.77) \quad \left( D^{F, \phi_1(F_1^\perp) \otimes \phi_2(F_2^\perp), \beta, \varepsilon} \right)^2 = -\Delta^{F, \phi_1(F_1^\perp) \otimes \phi_2(F_2^\perp), \beta, \varepsilon} + \frac{k^F}{4\beta^2} + O\left(\frac{1}{\beta} + \frac{\varepsilon^2}{\beta^2}\right).$$

**Proposition 1.9.** *If  $k^F > 0$  over  $M$ , then for any Pontrjagin classes  $p(F_1^\perp)$ ,  $p'(F_2^\perp)$  of  $F_1^\perp$ ,  $F_2^\perp$  respectively, the following identity holds,*

$$(1.78) \quad \left\langle \widehat{A}(F) p(F_1^\perp) e(F_2^\perp) p'(F_2^\perp), [M] \right\rangle = 0.$$

*Proof.* Since  $k^F > 0$  over  $M$ , one can take  $\beta > 0$ ,  $\varepsilon > 0$  small enough so that the corresponding terms in the right hand side of (1.77) verifies that

$$(1.79) \quad \frac{k^F}{4\beta^2} + O\left(\frac{1}{\beta} + \frac{\varepsilon^2}{\beta^2}\right) > 0$$

over  $M$ . Since  $-\Delta^{F, \phi_1(F_1^\perp) \otimes \phi_2(F_2^\perp), \beta, \varepsilon}$  is nonnegative, by (1.68), (1.77) and (1.79), one gets

$$(1.80) \quad \text{ind} \left( D_+^{F, \phi_1(F_1^\perp) \otimes \phi_2(F_2^\perp), \beta, \varepsilon} \right) = 0.$$

From (1.74) and (1.80), we get

$$(1.81) \quad \left\langle \widehat{A}(F) \widehat{L}(F_1^\perp) \text{ch}(\phi_1(F_1^\perp)) e(F_2^\perp) \text{ch}(\phi_2(F_2^\perp)), [M] \right\rangle = 0.$$

Now as it is standard that any Pontrjagin class of  $F_1^\perp$  (resp.  $F_2^\perp$ ) can be expressed as a rational linear combination of the classes of the form  $\widehat{L}(F_1^\perp) \text{ch}(\phi_1(F_1^\perp))$  (resp.  $\text{ch}(\phi_2(F_2^\perp))$ ), one gets (1.78) from (1.81).  $\square$

**Remark 1.10.** Recall that  $F^\perp = F_1^\perp \oplus F_2^\perp$ . It is proved in [18, Theorem 2.6] that if the conditions in Proposition 1.9 hold, then  $\langle \widehat{A}(F) p(F^\perp), [M] \rangle = 0$ . Here if one changes the  $\mathbf{Z}_2$ -grading in the definition of the sub-Dirac operator by replacing  $\widetilde{\tau}(F_2^\perp)$  in (1.62) by  $\tau(F_2^\perp, g^{F_2^\perp})$ , then one can prove that under the same condition as in Proposition 1.9,

$$(1.82) \quad \left\langle \widehat{A}(F) p(F_1^\perp) p'(F_2^\perp), [M] \right\rangle = 0$$

for any Pontrjagin classes  $p(F_1^\perp)$ ,  $p'(F_2^\perp)$  of  $F_1^\perp$ ,  $F_2^\perp$ .

**Remark 1.11.** Formulas (1.78) and (1.82) hold indeed without Condition (C) in Definition 1.2. This can be checked if we set  $\varepsilon = \sqrt{\beta}$ .

## 2. CONNES TYPE FIBRATION AND VANISHING THEOREMS

In this Section we prove Theorems 0.1, 0.4 and 0.5. We will make use of the Connes fibration which has indeed played an essential role in Connes' original proof given in [6].

This Section is organized as follows. In Section 2.1, we introduce what we call a Connes type fibration over a foliation, inspired by Connes' original construction. In Section 2.2 we introduce a coordinate system near the embedded submanifold from the original foliation into the Connes type foliation. In Section 2.3, we give an adiabatic limit estimate of the sub-Dirac operator on the Connes type fibration. In Section 2.4, we embed the smooth sections over the embedded submanifold to the space of smooth sections, having compact support near the embedded submanifold, on the Connes type fibration. In Section 2.5, we state an important estimate result which will be proved in Sections 2.6-2.8. In Section 2.9, we specify the above estimate result on a doubled Connes fibration and prove a key positivity result. In Sections 2.10-2.13, we complete the proofs of Theorems 0.1, 0.4 and 0.5 respectively.

**2.1. Connes type fibrations.** Let  $(M, F)$  be a compact foliation, where  $F$  is an integrable subbundle of the tangent vector bundle  $TM$  of a closed manifold  $M$ . We make the assumption that  $TM, F$  are oriented, then  $TM/F$  is also oriented. We further assume that  $F$  is spin and carries a fixed spin structure.

Inspired by the original construction in [6, Section 5], which will be recalled later in Section 2.9, we make the following definition.

**Definition 2.1.** By a Connes type fibration over  $(M, F)$  we mean a fibration  $\pi : \mathcal{M} \rightarrow M$  such that (i) there exists a splitting  $T\mathcal{M} = T^V\mathcal{M} \oplus T^H\mathcal{M}$ , where  $T^V\mathcal{M}$  is the vertical tangent bundle of the fibration, such that  $F$  lifts to an integrable subbundle  $\mathcal{F} \subset T^H\mathcal{M}$ ; (ii) if we denote  $T^V\mathcal{M} = \mathcal{F}_2^\perp$ , then there exists a splitting  $T^H\mathcal{M} = \mathcal{F} \oplus \mathcal{F}_1^\perp$  as well as Euclidean metrics  $g^{\mathcal{F}_1^\perp}$ ,  $g^{\mathcal{F}_2^\perp}$  on  $\mathcal{F}_1^\perp$ ,  $\mathcal{F}_2^\perp$  such that the foliation  $(\mathcal{M}, \mathcal{F})$  carries an associated almost isometric foliation structure in the sense of Section 1.1;<sup>5</sup> (iii) there exists a smooth (embedded) section  $s : M \hookrightarrow \mathcal{M}$ .

One of the specific features of a Connes type fibration is that since  $\mathcal{F}_2^\perp = T^V\mathcal{M}$  is the vertical tangent bundle of a fibration, the following identity holds:

$$(2.1) \quad [U, V] \in \Gamma(\mathcal{F}_2^\perp) \quad \text{for} \quad U, V \in \Gamma(\mathcal{F}_2^\perp).$$

That is, Condition (C) in Definition 1.2 holds for  $(\mathcal{M}, \mathcal{F})$ . Combining with (1.1) and the second identity in (1.5), one sees that  $\mathcal{F} \oplus \mathcal{F}_2^\perp$  is also an integrable subbundle of  $T\mathcal{M}$ .

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<sup>5</sup>We will use notation similarly as in Section 1, only in that in dealing with the Connes type fibration, we decorate the original notation in a "cal" way.

Let  $g^F$  be a Euclidean metric on  $F$ , then it lifts to a Euclidean metric  $g^{\mathcal{F}}$  on  $\mathcal{F}$ . Let  $g^{T\mathcal{M}}$  be the Riemannian metric on  $T\mathcal{M}$  defined by the following orthogonal splitting,

$$(2.2) \quad T\mathcal{M} = \mathcal{F} \oplus \mathcal{F}_1^\perp \oplus \mathcal{F}_2^\perp, \quad g^{T\mathcal{M}} = g^{\mathcal{F}} \oplus g^{\mathcal{F}_1^\perp} \oplus g^{\mathcal{F}_2^\perp}.$$

For any  $\beta > 0$ ,  $\varepsilon > 0$ , let  $g_{\beta,\varepsilon}^{T\mathcal{M}}$  be the Riemannian metric on  $T\mathcal{M}$  defined as in (1.9). By (1.8), (1.9) and (2.1), the following identity holds for the Connes type fibration,

$$(2.3) \quad \nabla^{\mathcal{F}_2^\perp, \beta, \varepsilon} = \nabla^{\mathcal{F}_2^\perp}.$$

Equivalently, for any  $X \in T\mathcal{M}$  and  $U, V \in \Gamma(\mathcal{F}_2^\perp)$ , one has  $\langle \nabla_X^{\mathcal{F}_2^\perp, \beta, \varepsilon} U, V \rangle = \langle \nabla_X^{\mathcal{F}_2^\perp} U, V \rangle$ .

We assume that  $\mathcal{F}_2^\perp = T^V\mathcal{M}$  is oriented, then  $T\mathcal{M}$  is also oriented in view of (2.2).

**2.2. A coordinate system near  $s(M)$ .** Let  $s(M) \subset \mathcal{M}$  be the image of the embedded section  $s : M \hookrightarrow \mathcal{M}$ . Consider the induced fibration  $s \circ \pi : \mathcal{M} \rightarrow s(M)$ . In what follows, for any  $x \in s(M)$ , we will denote the fiber  $\mathcal{M}_{\pi(x)}$  simply by  $\mathcal{M}_x$ .

For any  $x \in s(M)$ ,  $Z \in T_x\mathcal{M}_x = \mathcal{F}_2^\perp|_x$  with  $|Z|$  sufficiently small, let  $\exp^{\mathcal{M}_x}(tZ)$  be the geodesic in  $\mathcal{M}_x$  such that  $\exp^{\mathcal{M}_x}(0) = x$ ,  $\frac{d \exp^{\mathcal{M}_x}(tZ)}{dt}|_{t=0} = Z$ .

Let  $\alpha > 0$  be sufficiently small so that the above exponential map induces a diffeomorphism from  $U_\alpha(\mathcal{F}_2^\perp) = \{(x, Z) : x \in s(M), Z \in \mathcal{F}_2^\perp|_x, |Z| < \alpha\}$  to an open neighborhood  $U_\alpha \subset \mathcal{M}$  of  $s(M)$ . In what follows, we identify  $(x, Z) \in \mathcal{F}_2^\perp|_x$ ,  $|Z| < \alpha$ , with the corresponding point in  $\mathcal{M}_x$ . In particular,  $(x, 0)$  is identified with  $x$ . We also denote the geodesic  $\exp^{\mathcal{M}_x}(tZ)$  by  $tZ$ .

On  $U_\alpha(\mathcal{F}_2^\perp)$ , the volume form  $dv_{\mathcal{M}}$  can be written as

$$(2.4) \quad dv_{\mathcal{M}}(x, Z) = k(x, Z) dv_{\mathcal{F}_2^\perp|_x}(Z) dv_{s(M)}(x),$$

where  $dv_{\mathcal{F}_2^\perp|_x}$  is the volume form on  $\mathcal{F}_2^\perp|_x = \mathcal{F}_2^\perp|_x$  which in turn determines the corresponding volume form on  $\mathcal{M}_x \cap U_\alpha(\mathcal{F}_2^\perp)$ ,  $dv_{s(M)}$  is the volume form on  $s(M)$  with respect to the restricted metric, and  $k(x, Z) > 0$  is the function determined by (2.4).<sup>6</sup>

In what follows, we will also denote  $dv_{\mathcal{F}_2^\perp|_x}$  by  $dv_{\mathcal{M}_x}$ .

**2.3. Adiabatic limit near  $s(M)$ .** For simplicity, we assume that  $q = \text{rk}(\mathcal{F})$  and  $q_1 = \text{rk}(\mathcal{F}_1^\perp)$  are divisible by 8. Then all the spinor bundles and exterior algebras have real structures. So we can work on the category of real spaces.

Recall that for  $\beta > 0$  and  $\varepsilon > 0$ ,  $g_{\beta,\varepsilon}^{T\mathcal{M}}$  is the Riemannian metric on  $T\mathcal{M}$  defined by

$$(2.5) \quad g_{\beta,\varepsilon}^{T\mathcal{M}} = \beta^2 g^{\mathcal{F}} \oplus \frac{1}{\varepsilon^2} g^{\mathcal{F}_1^\perp} \oplus g^{\mathcal{F}_2^\perp},$$

and that  $D^{\mathcal{F}, \phi_1(\mathcal{F}_1^\perp), \beta, \varepsilon}$  is the sub-Dirac operator constructed in (1.67) with respect to  $g_{\beta,\varepsilon}^{T\mathcal{M}}$ .<sup>7</sup> By (2.5) one has

$$(2.6) \quad dv_{(T\mathcal{M}, g_{\beta,\varepsilon}^{T\mathcal{M}})} = \frac{\beta^q}{\varepsilon^{q_1}} dv_{(T\mathcal{M}, g^{T\mathcal{M}})}.$$

<sup>6</sup>As  $\mathcal{F}_2^\perp|_{s(M)}$  need not be orthogonal to  $Ts(M)$ ,  $k(x, 0)$  need not equal to 1 (compare with [4, (8.22)]).

<sup>7</sup>In this section, we will not consider the twisted bundle  $\phi_2(\mathcal{F}_2^\perp)$ , as it does not affect the final result.

For simplicity, from now on, by  $L^2$ -norms we will mean the  $L^2$ -norms with respect to the volume form  $dv_{(T\mathcal{M}, g^{T\mathcal{M}})}$ , i.e., for any  $s \in \Gamma(W(\mathcal{F}, \mathcal{F}_1^\perp, \mathcal{F}_2^\perp) \otimes \phi_1(\mathcal{F}_1^\perp))$  with compact support, one has

$$(2.7) \quad \|s\|_0^2 := \int_{\mathcal{M}} \langle s, s \rangle_{\beta, \varepsilon} dv_{(T\mathcal{M}, g^{T\mathcal{M}})},$$

where the subscripts “ $\beta, \varepsilon$ ” indicate that the pointwise inner product is induced from  $g_{\beta, \varepsilon}^{T\mathcal{M}}$ .

From (2.6) and (2.7), one sees that the operators which are formally self-adjoint with respect to the usual  $L^2$ -norm, which is associated with the volume form  $dv_{(T\mathcal{M}, g_{\beta, \varepsilon}^{T\mathcal{M}})}$ , is still formally self-adjoint with respect to the  $L^2$ -norm defined in (2.7).

By (1.77), one knows that when  $\beta, \varepsilon > 0$  are sufficiently small, the following identity holds on  $U_\alpha(\mathcal{F}_2^\perp)$ :

$$(2.8) \quad \left( D^{\mathcal{F}, \phi_1(\mathcal{F}_1^\perp), \beta, \varepsilon} \right)^2 = -\Delta^{\mathcal{F}, \phi_1(\mathcal{F}_1^\perp), \beta, \varepsilon} + \frac{k^{\mathcal{F}}}{4\beta^2} + O\left(\frac{1}{\beta} + \frac{\varepsilon^2}{\beta^2}\right).$$

Let  $h_1, \dots, h_{\dim \mathcal{M}}$  be an oriented orthonormal basis of  $(T\mathcal{M}, g_{\beta, \varepsilon}^{T\mathcal{M}})$ . Then for any  $s \in \Gamma(W(\mathcal{F}, \mathcal{F}_1^\perp, \mathcal{F}_2^\perp) \otimes \phi_1(\mathcal{F}_1^\perp))$  having compact support, the following identity holds:

$$(2.9) \quad \left\langle -\Delta^{\mathcal{F}, \phi_1(\mathcal{F}_1^\perp), \beta, \varepsilon} s, s \right\rangle = \sum_{i=1}^{\dim \mathcal{M}} \left\| \nabla_{h_i}^{\mathcal{F}, \phi_1(\mathcal{F}_1^\perp), \beta, \varepsilon} s \right\|_0^2.$$

On the other hand, for any  $s \in \Gamma((S(\mathcal{F}) \hat{\otimes} \Lambda^*(\mathcal{F}_1^\perp) \otimes \phi_1(\mathcal{F}_1^\perp))|_{s(M)})$ , similarly as in (2.7), we define its  $L^2$ -norm by

$$(2.10) \quad \|s\|_0^2 := \int_{s(M)} \langle s, s \rangle_{\beta, \varepsilon} dv_{(Ts(M), g^{Ts(M)})}.$$

In what follows, we will also denote  $dv_{(T\mathcal{M}, g^{T\mathcal{M}})}$ ,  $dv_{(Ts(M), g^{Ts(M)})}$  by  $dv_{\mathcal{M}}$ ,  $dv_{s(M)}$ .

**2.4. An embedding from sections on  $s(M)$  to sections on  $\mathcal{M}$ .** Recall that  $\Lambda^*(\mathcal{F}_2^\perp) = \bigoplus_{i=0}^{\text{rk}(\mathcal{F}_2^\perp)} \Lambda^i(\mathcal{F}_2^\perp)$ , with  $\Lambda^0(\mathcal{F}_2^\perp) = \mathbf{R}$ . Let

$$(2.11) \quad Q : \Lambda^*(\mathcal{F}_2^\perp) \rightarrow \Lambda^0(\mathcal{F}_2^\perp) = \mathbf{R}$$

denote the corresponding orthogonal projection. Let

$$(2.12) \quad i_Q : \Lambda^0(\mathcal{F}_2^\perp) \hookrightarrow \Lambda^*(\mathcal{F}_2^\perp)$$

denote the canonical inclusion. In view of (1.61) and (1.64), the projection  $Q$  and the embedding  $i_Q$  induce the following canonical orthogonal projection and embedding, which we will denote by the same notation,

$$(2.13) \quad Q : W(\mathcal{F}, \mathcal{F}_1^\perp, \mathcal{F}_2^\perp) \otimes \phi_1(\mathcal{F}_1^\perp) \rightarrow S(\mathcal{F}) \hat{\otimes} \Lambda^*(\mathcal{F}_1^\perp) \otimes \phi_1(\mathcal{F}_1^\perp),$$

$$(2.14) \quad i_Q : S(\mathcal{F}) \hat{\otimes} \Lambda^*(\mathcal{F}_1^\perp) \otimes \phi_1(\mathcal{F}_1^\perp) \hookrightarrow W(\mathcal{F}, \mathcal{F}_1^\perp, \mathcal{F}_2^\perp) \otimes \phi_1(\mathcal{F}_1^\perp).$$

Let  ${}^Q\nabla^{\mathcal{F}, \phi_1(\mathcal{F}_1^\perp), \beta, \varepsilon}$  be the induced connection on  $S(\mathcal{F}) \hat{\otimes} \Lambda^*(\mathcal{F}_1^\perp) \otimes \phi_1(\mathcal{F}_1^\perp)$  defined by

$$(2.15) \quad {}^Q\nabla^{\mathcal{F}, \phi_1(\mathcal{F}_1^\perp), \beta, \varepsilon} = Q\nabla^{\mathcal{F}, \phi_1(\mathcal{F}_1^\perp), \beta, \varepsilon} i_Q.$$

Let  $\sigma \in \Gamma((S(\mathcal{F}) \hat{\otimes} \Lambda^*(\mathcal{F}_1^\perp) \otimes \phi_1(\mathcal{F}_1^\perp))|_{s(M)})$ . For any  $(x, Z) \in U_\alpha(\mathcal{F}_2^\perp)$ , let  $\tau\sigma(x, Z) \in (S(\mathcal{F}) \hat{\otimes} \Lambda^*(\mathcal{F}_1^\perp) \otimes \phi_1(\mathcal{F}_1^\perp))|_{(x, Z)}$  be the parallel transport of  $\sigma(x)$  along the geodesic  $(x, tZ)$ ,  $0 \leq t \leq 1$ , with respect to the connection  ${}^Q\nabla^{\mathcal{F}, \phi_1(\mathcal{F}_1^\perp), \beta, \varepsilon}$ .

Let  $\gamma$  be a smooth function on  $\mathbf{R}$  such that  $\gamma(b) = 1$  if  $b \leq \frac{\alpha}{3}$ , while  $\gamma(b) = 0$  if  $b \geq \frac{2\alpha}{3}$ .

For  $T > 0$ ,  $x \in s(M)$ , set

$$(2.16) \quad \alpha_T(x) = \int_{\mathcal{M}_x} \exp(-T|Z|^2) \gamma^2(|Z|) dv_{\mathcal{M}_x}(Z).$$

Clearly,  $\alpha_T(x)$  is constant on  $s(M)$ , which we will denote by  $\alpha_T$ .

Inspired by [4, Definition 9.4], for  $T > 0$ , let

$$J_{T, \beta, \varepsilon} : \Gamma((S(\mathcal{F}) \hat{\otimes} \Lambda^*(\mathcal{F}_1^\perp) \otimes \phi_1(\mathcal{F}_1^\perp))|_{s(M)}) \longrightarrow \Gamma(W(\mathcal{F}, \mathcal{F}_1^\perp, \mathcal{F}_2^\perp) \otimes \phi_1(\mathcal{F}_1^\perp))$$

be the embedding defined by

$$(2.17) \quad J_{T, \beta, \varepsilon} : \sigma \mapsto (J_{T, \beta, \varepsilon} \sigma)(x, Z) = (k(x, Z) \alpha_T)^{-\frac{1}{2}} \gamma(|Z|) \exp\left(-\frac{T|Z|^2}{2}\right) i_Q(\tau\sigma(x, Z)).$$

By the definition of  $\gamma$ , one sees that  $J_{T, \beta, \varepsilon}$  is well-defined. Moreover, in view of (2.4), (2.7), (2.10), (2.16) and (2.17), one sees that  $J_{T, \beta, \varepsilon}$  is an isometric embedding.

Clearly, any  $J_{T, \beta, \varepsilon} \sigma$  has compact support in  $\mathcal{M}_{2\alpha/3}$ . Let  $E_{T, \beta, \varepsilon}$  denote the image of  $\Gamma((S(\mathcal{F}) \hat{\otimes} \Lambda^*(\mathcal{F}_1^\perp) \otimes \phi_1(\mathcal{F}_1^\perp))|_{s(M)})$  under  $J_{T, \beta, \varepsilon}$ . Let  $p_{T, \beta, \varepsilon}$  denote the orthogonal projection from the  $L^2$ -completion of  $\Gamma((W(\mathcal{F}, \mathcal{F}_1^\perp, \mathcal{F}_2^\perp) \otimes \phi_1(\mathcal{F}_1^\perp))|_{U_\alpha(\mathcal{F}_2^\perp)})$  to the  $L^2$ -completion of  $E_{T, \beta, \varepsilon}$  (we will also denote this  $L^2$ -completion by  $E_{T, \beta, \varepsilon}$ ).

**2.5. An estimate result for  $\|p_{T, \beta, \varepsilon} D^{\mathcal{F}, \phi_1(\mathcal{F}_1^\perp), \beta, \varepsilon} p_{T, \beta, \varepsilon}\|_0^2$ .** Let  $f_1, \dots, f_{q+q_1}$  be an oriented orthonormal basis of  $(\mathcal{F} \oplus \mathcal{F}_1^\perp)|_{s(M)}$  with respect to  $g^{\mathcal{F}} \oplus g^{\mathcal{F}_1^\perp}$ , where  $f_1, \dots, f_q$  is an oriented orthonormal basis of  $\mathcal{F}|_{s(M)}$  and thus  $f_{q+1}, \dots, f_{q+q_1}$  is an oriented orthonormal basis of  $\mathcal{F}_1^\perp|_{s(M)}$ . Let  $e_1, \dots, e_{q_2}$  be an oriented orthonormal basis of  $\mathcal{F}_2^\perp|_{s(M)}$  with respect to  $g^{\mathcal{F}_2^\perp}$ .

For any  $f \in \mathcal{F}|_{s(M)} \oplus \mathcal{F}_1^\perp|_{s(M)}$  (resp.  $e \in \mathcal{F}_2^\perp|_{s(M)}$ ), let  $\tau f \in \Gamma(\mathcal{F} \oplus \mathcal{F}_1^\perp)$  (resp.  $\tau e \in \Gamma(\mathcal{F}_2^\perp)$ ) be such that for any  $(x, Z) \in U_\alpha(\mathcal{F}_2^\perp)$ ,  $\tau f(x, Z)$  (resp.  $\tau e(x, Z)$ ) is the parallel transport of  $f_x$  (resp.  $e_x$ ) along the geodesic  $(x, tZ)$ ,  $0 \leq t \leq 1$ , with respect to the connection  $(p + p_1^\perp) \nabla^{T\mathcal{M}, \beta, \varepsilon} (p + p_1^\perp)$  (resp.  $\nabla^{\mathcal{F}_2^\perp, \beta, \varepsilon} = \nabla^{\mathcal{F}_2^\perp}$ ).

Clearly,  $\beta^{-1} \tau f_i$  ( $1 \leq i \leq q$ ),  $\varepsilon \tau f_j$  ( $q+1 \leq j \leq q+q_1$ ) and  $\tau e_k$  ( $1 \leq k \leq q_2$ ) form an orthonormal basis of  $(T\mathcal{M}, g_{\beta, \varepsilon}^{T\mathcal{M}})$ .

Let  $c_{\beta, \varepsilon}(\cdot)$  be the Clifford action associated to  $g_{\beta, \varepsilon}^{T\mathcal{M}}$ . For any  $X, Y \in T\mathcal{M}$ , one has

$$(2.18) \quad c_{\beta, \varepsilon}(X) c_{\beta, \varepsilon}(Y) + c_{\beta, \varepsilon}(Y) c_{\beta, \varepsilon}(X) = -2\langle X, Y \rangle_{g_{\beta, \varepsilon}^{T\mathcal{M}}}.$$

By (1.67), one has

$$(2.19) \quad \begin{aligned} D^{\mathcal{F}, \phi_1(\mathcal{F}_1^\perp), \beta, \varepsilon} &= \beta^{-1} \sum_{i=1}^q c_{\beta, \varepsilon}(\beta^{-1} \tau f_i) \nabla_{\tau f_i}^{\mathcal{F}, \phi_1(\mathcal{F}_1^\perp), \beta, \varepsilon} + \varepsilon \sum_{i=q+1}^{q+q_1} c_{\beta, \varepsilon}(\varepsilon \tau f_i) \nabla_{\tau f_i}^{\mathcal{F}, \phi_1(\mathcal{F}_1^\perp), \beta, \varepsilon} \\ &\quad + \sum_{s=1}^{q_2} c_{\beta, \varepsilon}(\tau e_s) \nabla_{\tau e_s}^{\mathcal{F}, \phi_1(\mathcal{F}_1^\perp), \beta, \varepsilon}. \end{aligned}$$

From now on, we make the assumption that the leafwise scalar curvature  $k^F$  of  $g^F$  verifies that  $k^F \geq \eta > 0$  over  $M$ .

We state a key asymptotic estimate result for  $\|p_{T,\beta,\varepsilon} D^{\mathcal{F},\phi_1(\mathcal{F}_1^\perp),\beta,\varepsilon} p_{T,\beta,\varepsilon}\|_0^2$ , when  $T \rightarrow +\infty$  and  $\beta, \varepsilon > 0$  being small, as follows.

**Proposition 2.2.** *There exist  $C_1 > 0$ ,  $C_2 > 0$ ,  $0 < \delta$ ,  $\beta_0, \varepsilon_0 < 1$  and  $T_0 > 0$  such that for any  $0 < \beta \leq \beta_0$ ,  $0 < \varepsilon \leq \varepsilon_0$  and  $T \geq T_0$ , there exists  $C_{\beta,\varepsilon} > 0$  so that the following inequality holds for any  $\sigma \in \Gamma((S(\mathcal{F}) \hat{\otimes} \Lambda^*(\mathcal{F}_1^\perp) \otimes \phi_1(\mathcal{F}_1^\perp))|_{s(M)})$ :*

$$(2.20) \quad \left\| p_{T,\beta,\varepsilon} D^{\mathcal{F},\phi_1(\mathcal{F}_1^\perp),\beta,\varepsilon} J_{T,\beta,\varepsilon} \sigma \right\|_0^2 \geq \left( \frac{\eta}{4\beta^2} - C_1 \left( \frac{1}{\beta} + \frac{\varepsilon^\delta}{\beta^2} + \frac{\varepsilon}{\beta^4} \right) \right) \int_{s(M)} |\sigma|^2 dv_{s(M)} \\ - \frac{C_2}{\beta^2} \sum_{i=1}^q \sum_{t=q+1}^{q+q_1} \int_{s(M)} \left| p_1^\perp \nabla_{f_t}^{T\mathcal{M},\beta,\varepsilon} \left( \nabla_{f_i}^{\mathcal{F}_2^\perp} Z \right) \right|^2 \cdot |\sigma|^2 dv_{s(M)} \\ + \frac{\varepsilon^\delta}{8\beta^2} \sum_{k=1}^q \int_{s(M)} \left| Q \nabla_{f_k}^{\mathcal{F},\phi_1(\mathcal{F}_1^\perp),\beta,\varepsilon} (\tau\sigma) \right|^2 dv_{s(M)} + \frac{\varepsilon^2}{8} \sum_{k=q+1}^{q+q_1} \int_{s(M)} \left| Q \nabla_{f_k}^{\mathcal{F},\phi_1(\mathcal{F}_1^\perp),\beta,\varepsilon} (\tau\sigma) \right|^2 dv_{s(M)} \\ - \frac{C_{\beta,\varepsilon}}{\sqrt{T}} \int_{s(M)} \left( |\sigma|^2 + \sum_{k=1}^{q+q_1} \left| Q \nabla_{f_k}^{\mathcal{F},\phi_1(\mathcal{F}_1^\perp),\beta,\varepsilon} (\tau\sigma) \right|^2 \right) dv_{s(M)},$$

where, in the second term in the right hand side,  $Z$  denotes the canonical section of  $\mathcal{F}_2^\perp|_{U_\alpha(\mathcal{F}_2^\perp)}$  which takes value  $\sum_{i=1}^{q_2} z_i \tau e_i$  over the point  $(x, Z = \sum_{i=1}^{q_2} z_i e_i)$ .

**Remark 2.3.** Proposition 2.2 holds on any Connes type fibration over  $(M, F)$ . However, the second term in the right hand side of (2.20) becomes an obstruction to get the positivity of the term in the left hand side of (2.20), which is our ultimate goal. Indeed, this obstruction comes from the fact that  $\mathcal{F}|_{s(M)}$  need not be included in  $Ts(M)$ . Thus if one writes  $Z = \sum_{i=1}^{q_2} z_i \tau e_i$  near  $s(M)$ , then one need not have  $f_i(z_j) = 0$  on  $s(M)$ .<sup>8</sup> This is the key difference with respect to the situation considered in [4, Chapters 8 and 9].<sup>9</sup>

The basic idea of the proof of Proposition 2.2 is very natural. Indeed, since  $p_{T,\beta,\varepsilon} : L^2((W(\mathcal{F}, \mathcal{F}_1^\perp, \mathcal{F}_2^\perp) \otimes \phi_1(\mathcal{F}_1^\perp))|_{U_\alpha(\mathcal{F}_2^\perp)}) \rightarrow E_{T,\beta,\varepsilon}$  is an orthogonal projection, for any  $\sigma \in \Gamma((S(\mathcal{F}) \hat{\otimes} \Lambda^*(\mathcal{F}_1^\perp) \otimes \phi_1(\mathcal{F}_1^\perp))|_{s(M)})$ , one has

$$(2.21) \quad \left\| p_{T,\beta,\varepsilon} D^{\mathcal{F},\phi_1(\mathcal{F}_1^\perp),\beta,\varepsilon} J_{T,\beta,\varepsilon} \sigma \right\|_0^2 = \left\| D^{\mathcal{F},\phi_1(\mathcal{F}_1^\perp),\beta,\varepsilon} J_{T,\beta,\varepsilon} \sigma \right\|_0^2 \\ - \left\| (1 - p_{T,\beta,\varepsilon}) D^{\mathcal{F},\phi_1(\mathcal{F}_1^\perp),\beta,\varepsilon} J_{T,\beta,\varepsilon} \sigma \right\|_0^2.$$

In view of (2.6) and (2.7), the operator  $D^{\mathcal{F},\phi_1(\mathcal{F}_1^\perp),\beta,\varepsilon}$  is formally self-adjoint with respect to the  $L^2$ -norm in (2.21). Thus, the first term in the right hand side of (2.21) can be

<sup>8</sup>Indeed, one has on  $s(M)$  that  $p_1^\perp \nabla_{f_t}^{T\mathcal{M},\beta,\varepsilon} (\nabla_{f_i}^{\mathcal{F}_2^\perp} Z) = \sum_{j=1}^{q_2} f_i(z_j) p_1^\perp \nabla_{f_t}^{T\mathcal{M},\beta,\varepsilon} \tau e_j$ .

<sup>9</sup>One may also want to consider the splitting  $T\mathcal{M}|_{s(M)} = Ts(M) \oplus N$  on  $s(M)$ , similarly as in [4], where  $N$  is the normal bundle orthogonal to  $Ts(M)$  in  $T\mathcal{M}|_{s(M)}$ . However, then this splitting would depend on  $\beta$  and  $\varepsilon$  which would cause other kinds of troubles.

estimated by using (2.8) and (2.9). So we need to estimate the second term in the right hand side of (2.21), to make it as small as possible.

In what follows, for brevity, we also write  $\widetilde{\nabla}^{\mathcal{F},\beta,\varepsilon}$  for  $\nabla^{\mathcal{F},\phi_1(\mathcal{F}_1^\perp),\beta,\varepsilon}$ . By (2.19), one has

$$\begin{aligned}
(2.22) \quad & \left\| (1 - p_{T,\beta,\varepsilon}) D^{\mathcal{F},\phi_1(\mathcal{F}_1^\perp),\beta,\varepsilon} J_{T,\beta,\varepsilon} \sigma \right\|_0^2 = \sum_{i=1}^q \left\| (1 - p_{T,\beta,\varepsilon}) c_{\beta,\varepsilon} (\beta^{-1} \tau f_i) \widetilde{\nabla}_{\beta^{-1} \tau f_i}^{\mathcal{F},\beta,\varepsilon} J_{T,\beta,\varepsilon} \sigma \right\|_0^2 \\
& + \sum_{i=q+1}^{q+q_1} \left\| (1 - p_{T,\beta,\varepsilon}) c_{\beta,\varepsilon} (\varepsilon \tau f_i) \widetilde{\nabla}_{\varepsilon \tau f_i}^{\mathcal{F},\beta,\varepsilon} J_{T,\beta,\varepsilon} \sigma \right\|_0^2 + \sum_{i=1}^{q_2} \left\| (1 - p_{T,\beta,\varepsilon}) c_{\beta,\varepsilon} (\tau e_i) \widetilde{\nabla}_{\tau e_i}^{\mathcal{F},\beta,\varepsilon} J_{T,\beta,\varepsilon} \sigma \right\|_0^2 \\
& + \sum_{i \neq j, 1 \leq i, j \leq q} \left\langle (1 - p_{T,\beta,\varepsilon}) c_{\beta,\varepsilon} (\beta^{-1} \tau f_i) \widetilde{\nabla}_{\beta^{-1} \tau f_i}^{\mathcal{F},\beta,\varepsilon} J_{T,\beta,\varepsilon} \sigma, c_{\beta,\varepsilon} (\beta^{-1} \tau f_j) \widetilde{\nabla}_{\beta^{-1} \tau f_j}^{\mathcal{F},\beta,\varepsilon} J_{T,\beta,\varepsilon} \sigma \right\rangle \\
& + \sum_{i \neq j, q+1 \leq i, j \leq q+q_1} \left\langle (1 - p_{T,\beta,\varepsilon}) c_{\beta,\varepsilon} (\varepsilon \tau f_i) \widetilde{\nabla}_{\varepsilon \tau f_i}^{\mathcal{F},\beta,\varepsilon} J_{T,\beta,\varepsilon} \sigma, c_{\beta,\varepsilon} (\varepsilon \tau f_j) \widetilde{\nabla}_{\varepsilon \tau f_j}^{\mathcal{F},\beta,\varepsilon} J_{T,\beta,\varepsilon} \sigma \right\rangle \\
& + \sum_{i \neq j, 1 \leq i, j \leq q_2} \left\langle (1 - p_{T,\beta,\varepsilon}) c_{\beta,\varepsilon} (\tau e_i) \widetilde{\nabla}_{\tau e_i}^{\mathcal{F},\beta,\varepsilon} J_{T,\beta,\varepsilon} \sigma, c_{\beta,\varepsilon} (\tau e_j) \widetilde{\nabla}_{\tau e_j}^{\mathcal{F},\beta,\varepsilon} J_{T,\beta,\varepsilon} \sigma \right\rangle \\
& + 2 \sum_{i=1}^q \sum_{j=q+1}^{q+q_1} \left\langle (1 - p_{T,\beta,\varepsilon}) c_{\beta,\varepsilon} (\beta^{-1} \tau f_i) \widetilde{\nabla}_{\beta^{-1} \tau f_i}^{\mathcal{F},\beta,\varepsilon} J_{T,\beta,\varepsilon} \sigma, c_{\beta,\varepsilon} (\varepsilon \tau f_j) \widetilde{\nabla}_{\varepsilon \tau f_j}^{\mathcal{F},\beta,\varepsilon} J_{T,\beta,\varepsilon} \sigma \right\rangle \\
& + 2 \sum_{i=1}^q \sum_{j=1}^{q_2} \left\langle (1 - p_{T,\beta,\varepsilon}) c_{\beta,\varepsilon} (\beta^{-1} \tau f_i) \widetilde{\nabla}_{\beta^{-1} \tau f_i}^{\mathcal{F},\beta,\varepsilon} J_{T,\beta,\varepsilon} \sigma, c_{\beta,\varepsilon} (\tau e_j) \widetilde{\nabla}_{\tau e_j}^{\mathcal{F},\beta,\varepsilon} J_{T,\beta,\varepsilon} \sigma \right\rangle \\
& + 2 \sum_{i=q+1}^{q+q_1} \sum_{j=1}^{q_2} \left\langle (1 - p_{T,\beta,\varepsilon}) c_{\beta,\varepsilon} (\varepsilon \tau f_i) \widetilde{\nabla}_{\varepsilon \tau f_i}^{\mathcal{F},\beta,\varepsilon} J_{T,\beta,\varepsilon} \sigma, c_{\beta,\varepsilon} (\tau e_j) \widetilde{\nabla}_{\tau e_j}^{\mathcal{F},\beta,\varepsilon} J_{T,\beta,\varepsilon} \sigma \right\rangle.
\end{aligned}$$

Naturally, we need to study the behaviour when  $T \rightarrow +\infty$  of each term in the right hand side of (2.22). By the Gaussian factor  $\exp(-T|Z|^2/2)$  in (2.17), one sees as in [4, Chapters 8 and 9] that when  $T \rightarrow +\infty$ , everything would localize onto  $s(M)$ . All one need is to take care the rescaling factors  $\beta, \varepsilon$  to make the estimate goes as desired. For this the geometric nature of the Connes type fibration plays essential roles

The fact that the right hand side of (2.22) has nine terms, with each term further splits into four terms in the process of estimation, partly explains the length of the computations, which are purely routine and elementary.

Proposition 2.2 will be applied to a specific Connes type fibration in Section 2.9 so that one can go further to get a positivity result for the left hand side of (2.20). The reader who wishes to see directly that positivity can skip temporarily the next three subsections where a detailed proof of Proposition 2.2 is given.

**2.6. Estimates of inner product terms in (2.22), Part I.** Before going on, we set a notational convention: in what follows, by  $O(|Z|^2)$  and  $O(\frac{1}{\sqrt{T}})$ , we will mean  $O_{\beta,\varepsilon}(|Z|^2)$  and  $O_{\beta,\varepsilon}(\frac{1}{\sqrt{T}})$ , i.e., the associated estimating constants may depend on  $\beta > 0$  and  $\varepsilon > 0$ . While for other  $O(\dots)$  terms, the corresponding estimating constants will not depend on  $\beta > 0$  and  $\varepsilon > 0$ , unless there appear the subscripts “ $\beta$ ” and/or “ $\varepsilon$ ” which will indicate that the corresponding estimating coefficient will depend on  $\beta$  and/or  $\varepsilon$ .

For brevity, let  $f_T$  be the smooth function on  $\mathcal{M}$  defined by

$$(2.23) \quad f_T(x, Z) = (k(x, Z)\alpha_T)^{-\frac{1}{2}} \gamma(|Z|) \exp\left(-\frac{T|Z|^2}{2}\right).$$

Then one can rewrite  $J_{T,\beta,\varepsilon}\sigma$  in (2.17) as

$$(2.24) \quad (J_{T,\beta,\varepsilon}\sigma)(x, Z) = f_T(x, Z)i_Q(\tau\sigma(x, Z)).$$

From now on, in case of no confusion, we will omit  $i_Q$ .

**Lemma 2.4.** (i) For any  $\sigma \in \Gamma((S(\mathcal{F})\widehat{\otimes}\Lambda^*(\mathcal{F}_1^\perp) \otimes \phi_1(\mathcal{F}_1^\perp))|_{s(M)})$  and any  $f \in C^\infty(\mathcal{M})$  with  $\text{Supp}(f) \subset U_\alpha(\mathcal{F}_2^\perp)$ , one has

$$(2.25) \quad (p_{T,\beta,\varepsilon}(f\tau\sigma))(x, Z) = \left( \int_{\mathcal{M}_x} f_T(x, Z')f(x, Z')k(x, Z')dv_{\mathcal{M}_x}(Z') \right) (J_{T,\beta,\varepsilon}\sigma)(x, Z);$$

(ii) For any  $u \in \Gamma(W(\mathcal{F}, \mathcal{F}_1^\perp, \mathcal{F}_2^\perp) \otimes \phi_1(\mathcal{F}_1^\perp))$  with  $\text{Supp}(u) \subset U_\alpha(\mathcal{F}_2^\perp)$ , one has

$$(2.26) \quad p_{T,\beta,\varepsilon}(f_T u) = J_{T,\beta,\varepsilon}((Qu)|_{s(M)}) + p_{T,\beta,\varepsilon}(O_{\beta,\varepsilon}(|Z|)).$$

*Proof.* Take any  $u \in \Gamma((W(\mathcal{F}, \mathcal{F}_1^\perp, \mathcal{F}_2^\perp) \otimes \phi_1(\mathcal{F}_1^\perp))|_{U_\alpha(\mathcal{F}_2^\perp)})$ . Then for any  $(x, Z) \in U_\alpha(\mathcal{F}_2^\perp)$ ,  $(Qu)|_{(x,Z)}$  determines a unique element  $u' \in (S(\mathcal{F})\widehat{\otimes}\Lambda^*(\mathcal{F}_1^\perp) \otimes \phi_1(\mathcal{F}_1^\perp))|_x$  such that  $(\tau u')|_{(x,Z)} = (Qu)|_{(x,Z)}$ . We denote this element by  $\tau^{-1}((Qu)|_{(x,Z)})$ .

Then one verifies easily that (compare with [4, (9.6) and (9.13)])

$$(2.27) \quad (p_{T,\beta,\varepsilon}u)(x, Z) = f_T(x, Z) \left( \tau \int_{\mathcal{M}_x} f_T(x, Z')k(x, Z')\tau^{-1}((Qu)|_{(x,Z')})dv_{\mathcal{M}_x}(Z') \right) (x, Z).$$

Formulas (2.25) and (2.26) follow from (2.27) easily.  $\square$

**Lemma 2.5.** For any  $X \in \Gamma((\mathcal{F} \oplus \mathcal{F}_1^\perp)|_{s(M)})$ , one has

$$(2.28) \quad p_{T,\beta,\varepsilon}c_{\beta,\varepsilon}(\tau X) = c_{\beta,\varepsilon}(\tau X)p_{T,\beta,\varepsilon}.$$

*Proof.* For any  $\sigma \in \Gamma((S(\mathcal{F})\widehat{\otimes}\Lambda^*(\mathcal{F}_1^\perp) \otimes \phi_1(\mathcal{F}_1^\perp))|_{s(M)})$  and  $X \in \Gamma((\mathcal{F} \oplus \mathcal{F}_1^\perp)|_{s(M)})$ , we claim that

$$(2.29) \quad c_{\beta,\varepsilon}(\tau X)\tau\sigma = \tau(c_{\beta,\varepsilon}(X)\sigma).$$

Indeed, it is easy to verify that

$$(2.30) \quad \begin{aligned} Q\widetilde{\nabla}_Z^{\mathcal{F},\beta,\varepsilon}(c_{\beta,\varepsilon}(\tau X)\tau\sigma) &= Q\left(c_{\beta,\varepsilon}\left(\nabla_Z^{T\mathcal{M},\beta,\varepsilon}(\tau X)\right)\tau\sigma\right) + c_{\beta,\varepsilon}(\tau X)Q\widetilde{\nabla}_Z^{\mathcal{F},\beta,\varepsilon}(\tau\sigma) \\ &= c_{\beta,\varepsilon}\left((p + p_1^\perp)\nabla_Z^{T\mathcal{M},\beta,\varepsilon}(\tau X)\right)\tau\sigma = 0. \end{aligned}$$

From (2.30), one sees that  $c_{\beta,\varepsilon}(\tau X)\tau\sigma$  is the parallel transport of  $(c_{\beta,\varepsilon}(\tau X)\tau\sigma)|_{s(M)} = c_{\beta,\varepsilon}(X)\sigma$ , from which (2.29) follows.



Now for any  $\sigma \in \Gamma((S(\mathcal{F}) \widehat{\otimes} \Lambda^*(\mathcal{F}_1^\perp) \otimes \phi_1(\mathcal{F}_1^\perp))|_{s(M)})$  and  $u \in \Gamma(W(\mathcal{F}, \mathcal{F}_1^\perp, \mathcal{F}_2^\perp) \otimes \phi_1(\mathcal{F}_1^\perp))$  with  $\text{Supp}(u) \subset U_\alpha(\mathcal{F}_2^\perp)$ , one verifies via (2.29) that

$$\begin{aligned} (2.31) \quad \langle p_{T,\beta,\varepsilon} c_{\beta,\varepsilon}(\tau X)u, J_{T,\beta,\varepsilon}\sigma \rangle &= \langle c_{\beta,\varepsilon}(\tau X)u, J_{T,\beta,\varepsilon}\sigma \rangle = -\langle u, c_{\beta,\varepsilon}(\tau X)J_{T,\beta,\varepsilon}\sigma \rangle \\ &= -\langle u, J_{T,\beta,\varepsilon}(c_{\beta,\varepsilon}(X)\sigma) \rangle = -\langle p_{T,\beta,\varepsilon}u, J_{T,\beta,\varepsilon}(c_{\beta,\varepsilon}(X)\sigma) \rangle = -\langle p_{T,\beta,\varepsilon}u, c_{\beta,\varepsilon}(\tau X)J_{T,\beta,\varepsilon}\sigma \rangle \\ &= \langle c_{\beta,\varepsilon}(\tau X)p_{T,\beta,\varepsilon}u, J_{T,\beta,\varepsilon}\sigma \rangle, \end{aligned}$$

from which (2.28) follows.  $\square$

For any  $X \in \Gamma((\mathcal{F} \oplus \mathcal{F}_1^\perp)|_{s(M)})$ , by (2.28), one finds

$$(2.32) \quad (1 - p_{T,\beta,\varepsilon}) c_{\beta,\varepsilon}(\tau X) = c_{\beta,\varepsilon}(\tau X) (1 - p_{T,\beta,\varepsilon}).$$

Let  $f'_i$ ,  $1 \leq i \leq q$  (resp.  $f'_j$ ,  $q+1 \leq j \leq q+q_1$ ) be an orthonormal basis of  $(\mathcal{F}, g^\mathcal{F})$  (resp.  $(\mathcal{F}_1^\perp, g^{\mathcal{F}_1^\perp})$ ) on  $U_\alpha(\mathcal{F}_2^\perp)$ , which does not depend on  $\beta$  and  $\varepsilon$ , and which satisfies  $f'_i|_{s(M)} = f_i$  (resp.  $f'_j|_{s(M)} = f_j$ ).

Without loss of generality, we assume that  $f'_1, \dots, f'_q$  are lifted from corresponding elements on  $M$ . That is, there is an orthonormal basis  $\widehat{f}_1, \dots, \widehat{f}_q$  of  $(F, g^F)$  such that

$$(2.33) \quad f'_i = \pi^* \widehat{f}_i, \quad 1 \leq i \leq q.$$

**Lemma 2.6.** *The following asymptotic formulas at  $(x, Z)$  with  $x \in s(M)$ ,  $Z \in \mathcal{M}_x$ , hold near  $s(M)$ : (i) if  $1 \leq i \leq q$ , then*

$$(2.34) \quad \tau f_i = f'_i + \sum_{m=q+1}^{q+q_1} O(\varepsilon^2 |Z|) f'_m + O(|Z|^2);$$

(ii) if  $q+1 \leq i \leq q+q_1$ , then

$$(2.35) \quad \tau f_i = f'_i + \sum_{j=1}^q O\left(\frac{|Z|}{\beta^2}\right) f'_j + \sum_{m=q+1}^{q+q_1} O(|Z|) f'_m + O(|Z|^2).$$

*Proof.* We write

$$(2.36) \quad \tau f_i = f'_i + \sum_{k=1}^{q+q_1} \langle \tau f_i - f'_i, f'_k \rangle f'_k.$$

Since

$$(2.37) \quad (p + p_1^\perp) \nabla_Z^{T\mathcal{M}, \beta, \varepsilon} (\tau f_i) = 0,$$

one has for  $1 \leq i, k \leq q$  that

$$\begin{aligned} (2.38) \quad \langle \tau f_i - f'_i, f'_k \rangle_{(x,Z)} &= Z \langle \tau f_i, f'_k \rangle_{(x,Z)} + O(|Z|^2) \\ &= \left\langle \tau f_i, \nabla_Z^{T\mathcal{M}, \beta, \varepsilon} f'_k \right\rangle_{(x,Z)} + O(|Z|^2) = \left\langle f_i, \nabla_Z^{T\mathcal{M}, \beta, \varepsilon} f'_k \right\rangle_x + O(|Z|^2), \end{aligned}$$

while for  $1 \leq i \leq q$ ,  $q+1 \leq k \leq q+q_1$ , one has, by (1.5), (1.8),

$$\begin{aligned} (2.39) \quad \langle \tau f_i - f'_i, f'_k \rangle_{(x,Z)} &= Z \langle \tau f_i, f'_k \rangle_{(x,Z)} + O(|Z|^2) \\ &= \beta^2 \varepsilon^2 \left\langle f_i, \nabla_Z^{T\mathcal{M}, \beta, \varepsilon} f'_k \right\rangle_x + O(|Z|^2) = O(\varepsilon^2 |Z|) + O(|Z|^2). \end{aligned}$$

Now by (2.33), one has that for any  $e \in \Gamma(\mathcal{F}_2^\perp)$  and  $1 \leq i \leq q$ ,

$$(2.40) \quad [e, f'_i] \in \Gamma(\mathcal{F}_2^\perp),$$

from which one verifies that for any  $e \in \Gamma(\mathcal{F}_2^\perp)$  and  $1 \leq i, k \leq q$ ,

$$(2.41) \quad \langle f'_i, \nabla_e^{T\mathcal{M}, \beta, \varepsilon} f'_k \rangle = \langle e, \nabla_{f'_i}^{T\mathcal{M}, \beta, \varepsilon} f'_k \rangle = 0.$$

From (2.36), (2.38), (2.39) and (2.41), one gets (2.34).

By proceeding as in (2.38), one sees that for  $q+1 \leq m \leq q+q_1$ ,  $1 \leq k \leq q$ ,

$$(2.42) \quad \begin{aligned} \langle \tau f_m - f'_m, f'_k \rangle_{(x, Z)} &= Z \langle \tau f_m, f'_k \rangle_{(x, Z)} + O(|Z|^2) \\ &= \frac{1}{\beta^2 \varepsilon^2} \left\langle f_m, \nabla_Z^{T\mathcal{M}, \beta, \varepsilon} f'_k \right\rangle_x + O(|Z|^2) = O\left(\frac{|Z|}{\beta^2}\right) + O(|Z|^2), \end{aligned}$$

while for  $q+1 \leq m, k \leq q+q_1$ , one has

$$(2.43) \quad \begin{aligned} \langle \tau f_m - f'_m, f'_k \rangle_{(x, Z)} &= Z \langle \tau f_m, f'_k \rangle_{(x, Z)} + O(|Z|^2) \\ &= \left\langle f_m, \nabla_Z^{T\mathcal{M}, \beta, \varepsilon} f'_k \right\rangle_x + O(|Z|^2) = O(|Z|) + O(|Z|^2). \end{aligned}$$

From (2.36), (2.42) and (2.43), one gets (2.35).  $\square$

**Lemma 2.7.** *There exists a constant  $C_{\beta, \varepsilon} > 0$  such that the following estimate holds near  $s(M)$  for  $|Z| \leq 2\alpha/3$ : for any  $\sigma \in \Gamma((S(\mathcal{F}) \hat{\otimes} \Lambda^*(\mathcal{F}_1^\perp) \otimes \phi_1(\mathcal{F}_1^\perp))|_{s(M)})$ , one has*

$$(2.44) \quad \begin{aligned} \sum_{i=1}^{q+q_1} \left| Q \nabla_{\tau f_i}^{\mathcal{F}, \phi_1(\mathcal{F}_1^\perp), \beta, \varepsilon}(\tau \sigma) \right|_{(x, Z)}^2 + \sum_{j=1}^{q_2} \left| Q \nabla_{\tau e_j}^{\mathcal{F}, \phi_1(\mathcal{F}_1^\perp), \beta, \varepsilon}(\tau \sigma) \right|_{(x, Z)}^2 \\ \leq C_{\beta, \varepsilon} \left( \sum_{i=1}^{q+q_1} \left| Q \nabla_{f_i}^{\mathcal{F}, \phi_1(\mathcal{F}_1^\perp), \beta, \varepsilon}(\tau \sigma) \right|_x^2 + |\sigma|_x^2 \right). \end{aligned}$$

*Proof.* For any  $X \in (T\mathcal{M})|_{s(M)}$  and  $\sigma, \sigma' \in \Gamma((S(\mathcal{F}) \hat{\otimes} \Lambda^*(\mathcal{F}_1^\perp) \otimes \phi_1(\mathcal{F}_1^\perp))|_{s(M)})$ , one verifies that,

$$(2.45) \quad \begin{aligned} \left\langle Q \nabla_{\tau X}^{\mathcal{F}, \phi_1(\mathcal{F}_1^\perp), \beta, \varepsilon}(\tau \sigma), \tau \sigma' \right\rangle_{\beta, \varepsilon} &= \tau X \langle \tau \sigma, \tau \sigma' \rangle_{\beta, \varepsilon} - \left\langle \tau \sigma, Q \nabla_{\tau X}^{\mathcal{F}, \phi_1(\mathcal{F}_1^\perp), \beta, \varepsilon}(\tau \sigma') \right\rangle_{\beta, \varepsilon} \\ &= \tau X \langle \sigma, \sigma' \rangle_{\beta, \varepsilon} - \left\langle \tau \sigma, Q \nabla_{\tau X}^{\mathcal{F}, \phi_1(\mathcal{F}_1^\perp), \beta, \varepsilon}(\tau \sigma') \right\rangle_{\beta, \varepsilon}. \end{aligned}$$

From (2.45) and let  $\sigma'$  run through the orthonormal basis of  $(S(\mathcal{F}) \hat{\otimes} \Lambda^*(\mathcal{F}_1^\perp) \otimes \phi_1(\mathcal{F}_1^\perp))|_{s(M)}$ , one obtains (2.44) easily.  $\square$

We now start to estimate the inner product terms in the right hand side of (2.22).

For any  $1 \leq i \leq q+q_1$ , we denote by  $\tilde{\tau} f_i$  the unit vector field (with respect to  $g_{\beta, \varepsilon}^{T\mathcal{M}}$ ) corresponding to  $\tau f_i$ , that is,

$$(2.46) \quad \tilde{\tau} f_i = \frac{\tau f_i}{|\tau f_i|_{\beta, \varepsilon}}.$$

Then, one has  $\tilde{\tau} f_i = \beta^{-1} \tau f_i$  if  $1 \leq i \leq q$ , while  $\tilde{\tau} f_i = \varepsilon \tau f_i$  if  $q+1 \leq i \leq q+q_1$ .

Let  $1 \leq i, j \leq q + q_1$  be such that  $i \neq j$ . By (2.32) one deduces that

$$\begin{aligned}
 (2.47) \quad & \left\langle (1 - p_{T,\beta,\varepsilon}) c_{\beta,\varepsilon} (\tilde{\tau} f_i) \tilde{\nabla}_{\tilde{\tau} f_i}^{\mathcal{F},\beta,\varepsilon} J_{T,\beta,\varepsilon} \sigma, (1 - p_{T,\beta,\varepsilon}) c_{\beta,\varepsilon} (\tilde{\tau} f_j) \tilde{\nabla}_{\tilde{\tau} f_j}^{\mathcal{F},\beta,\varepsilon} J_{T,\beta,\varepsilon} \sigma \right\rangle \\
 &= \langle c_{\beta,\varepsilon} (\tilde{\tau} f_i) (1 - p_{T,\beta,\varepsilon}) \tilde{\tau} f_i(f_T) \tau \sigma, c_{\beta,\varepsilon} (\tilde{\tau} f_j) (1 - p_{T,\beta,\varepsilon}) \tilde{\tau} f_j(f_T) \tau \sigma \rangle \\
 &+ \left\langle c_{\beta,\varepsilon} (\tilde{\tau} f_i) (1 - p_{T,\beta,\varepsilon}) \tilde{\tau} f_i(f_T) \tau \sigma, (1 - p_{T,\beta,\varepsilon}) c_{\beta,\varepsilon} (\tilde{\tau} f_j) f_T \tilde{\nabla}_{\tilde{\tau} f_j}^{\mathcal{F},\beta,\varepsilon} (\tau \sigma) \right\rangle \\
 &+ \left\langle (1 - p_{T,\beta,\varepsilon}) c_{\beta,\varepsilon} (\tilde{\tau} f_i) f_T \tilde{\nabla}_{\tilde{\tau} f_i}^{\mathcal{F},\beta,\varepsilon} (\tau \sigma), c_{\beta,\varepsilon} (\tilde{\tau} f_j) (1 - p_{T,\beta,\varepsilon}) \tilde{\tau} f_j(f_T) \tau \sigma \right\rangle \\
 &+ \left\langle (1 - p_{T,\beta,\varepsilon}) c_{\beta,\varepsilon} (\tilde{\tau} f_i) f_T \tilde{\nabla}_{\tilde{\tau} f_i}^{\mathcal{F},\beta,\varepsilon} (\tau \sigma), (1 - p_{T,\beta,\varepsilon}) c_{\beta,\varepsilon} (\tilde{\tau} f_j) f_T \tilde{\nabla}_{\tilde{\tau} f_j}^{\mathcal{F},\beta,\varepsilon} (\tau \sigma) \right\rangle.
 \end{aligned}$$

By (2.24) and (2.25), one has for any  $1 \leq i \leq q + q_1$ ,

$$(2.48) \quad (1 - p_{T,\beta,\varepsilon}) \tau f_i(f_T) \tau \sigma = \left( \tau f_i(f_T) - f_T \int_{\mathcal{M}_x} f_T \tau f_i(f_T) k dv_{\mathcal{M}_x} \right) \tau \sigma.$$

For any  $1 \leq i \leq q + q_1$ , set

$$(2.49) \quad \rho_{T,\beta,\varepsilon,i} = \tau f_i(f_T) - f_T \int_{\mathcal{M}_x} f_T \tau f_i(f_T) k dv_{\mathcal{M}_x}.$$

By (2.23), one has

$$(2.50) \quad \tau f_i(f_T)(x, Z) = \left( -\frac{\tau f_i(k) \gamma}{2k^{3/2} \sqrt{\alpha_T}} + \frac{\tau f_i(\gamma)}{k^{1/2} \sqrt{\alpha_T}} - \frac{T \tau f_i(|Z|^2) \gamma}{2k^{1/2} \sqrt{\alpha_T}} \right) \exp \left( -\frac{T|Z|^2}{2} \right).$$

Let  $Z = \sum_{i=1}^{q_2} z_i e_i$  in  $\mathcal{F}_2^\perp|_{s(M)}$ . Let  $a_{ik}^j \in C^\infty(s(M))$  be defined by

$$(2.51) \quad \tau f_i(z_j) = \tau f_i(z_j)|_{s(M)} + \sum_{k=1}^{q_2} a_{ij}^k z_k + O(|Z|^2).$$

By (2.23), (2.49)-(2.51) and Lemma 2.6, when  $T > 0$  is large enough, if  $1 \leq i \leq q$ ,

$$\begin{aligned}
 (2.52) \quad \rho_{T,\beta,\varepsilon,i}(x, Z) &= -\frac{T \tau f_i(|Z|^2)}{2} f_T(x, Z) + \frac{\tau f_i(\gamma)}{k^{1/2} \sqrt{\alpha_T}} (1 - \gamma) \exp \left( -\frac{T|Z|^2}{2} \right) \\
 &+ \frac{1}{2} \left( \sum_{j=1}^{q_2} a_{ij}^j + O(|Z|) + O(|Z|^2) + O \left( \frac{1}{\sqrt{T}} \right) \right) f_T(x, Z),
 \end{aligned}$$

while for  $q + 1 \leq i \leq q + q_1$ , one has

$$\begin{aligned}
 (2.53) \quad \rho_{T,\beta,\varepsilon,i}(x, Z) &= -\frac{T \tau f_i(|Z|^2)}{2} f_T(x, Z) + \frac{\tau f_i(\gamma)}{k^{1/2} \sqrt{\alpha_T}} (1 - \gamma) \exp \left( -\frac{T|Z|^2}{2} \right) \\
 &+ \frac{1}{2} \left( \sum_{j=1}^{q_2} a_{ij}^j + O \left( \frac{|Z|}{\beta^2} \right) + O(|Z|^2) + O \left( \frac{1}{\sqrt{T}} \right) \right) f_T(x, Z).
 \end{aligned}$$

We now start to estimate (2.47).

For the first term in the right hand side of (2.47), by (2.48) and (2.49), for  $i \neq j$ ,

$$\begin{aligned}
 (2.54) \quad & \langle c_{\beta,\varepsilon} (\tilde{\tau} f_i) (1 - p_{T,\beta,\varepsilon}) \tau f_i(f_T) \tau \sigma, c_{\beta,\varepsilon} (\tilde{\tau} f_j) (1 - p_{T,\beta,\varepsilon}) \tau f_j(f_T) \tau \sigma \rangle \\
 &= \langle c_{\beta,\varepsilon} (\tilde{\tau} f_i) c_{\beta,\varepsilon} (\tilde{\tau} f_j) \rho_{T,\beta,\varepsilon,i} \rho_{T,\beta,\varepsilon,j} \tau \sigma, \tau \sigma \rangle = 0,
 \end{aligned}$$

as  $c_{\beta,\varepsilon}(\tilde{\tau} f_i) c_{\beta,\varepsilon}(\tilde{\tau} f_j)$  is skew-adjoint.

For the second and the third terms in the right hand side of (2.47), by (2.32), one finds that for  $i \neq j$ ,

$$\begin{aligned}
(2.55) \quad & \left\langle c_{\beta,\varepsilon}(\tilde{\tau}f_i)(1-p_{T,\beta,\varepsilon})\tilde{\tau}f_i(f_T)\tau\sigma, (1-p_{T,\beta,\varepsilon})c_{\beta,\varepsilon}(\tilde{\tau}f_j)f_T\tilde{\nabla}_{\tilde{\tau}f_j}^{\mathcal{F},\beta,\varepsilon}(\tau\sigma) \right\rangle \\
&= \left\langle c_{\beta,\varepsilon}(\tilde{\tau}f_i)c_{\beta,\varepsilon}(\tilde{\tau}f_j)\tilde{\tau}f_i(f_T)\tau\sigma, (1-p_{T,\beta,\varepsilon})f_T\tilde{\nabla}_{\tilde{\tau}f_j}^{\mathcal{F},\beta,\varepsilon}(\tau\sigma) \right\rangle \\
&= \left\langle c_{\beta,\varepsilon}(\tilde{\tau}f_i)c_{\beta,\varepsilon}(\tilde{\tau}f_j)\tilde{\tau}f_i(f_T)f_T\tau\sigma, {}^Q\tilde{\nabla}_{\tilde{\tau}f_j}^{\mathcal{F},\beta,\varepsilon}(\tau\sigma) - \tau \left( {}^Q\tilde{\nabla}_{\tilde{\tau}f_j}^{\mathcal{F},\beta,\varepsilon}(\tau\sigma) \Big|_{s(M)} \right) \right\rangle \\
&- \left\langle c_{\beta,\varepsilon}(\tilde{\tau}f_i)c_{\beta,\varepsilon}(\tilde{\tau}f_j)f_Tp_{T,\beta,\varepsilon}(\tilde{\tau}f_i(f_T)\tau\sigma), {}^Q\tilde{\nabla}_{\tilde{\tau}f_j}^{\mathcal{F},\beta,\varepsilon}(\tau\sigma) - \tau \left( {}^Q\tilde{\nabla}_{\tilde{\tau}f_j}^{\mathcal{F},\beta,\varepsilon}(\tau\sigma) \Big|_{s(M)} \right) \right\rangle.
\end{aligned}$$

Since this term is more delicate to deal with than the other terms, we postpone it to the next subsection.

For the fourth term in the right hand side of (2.47), one first sees easily via (2.26) and (2.44) that when  $T > 0$  is large enough, for any  $x \in s(M)$ ,

$$\begin{aligned}
(2.56) \quad & \int_{\mathcal{M}_x} \left\langle (1-p_{T,\beta,\varepsilon})c_{\beta,\varepsilon}(\tilde{\tau}f_i)f_T\tilde{\nabla}_{\tilde{\tau}f_i}^{\mathcal{F},\beta,\varepsilon}(\tau\sigma), (1-p_{T,\beta,\varepsilon})c_{\beta,\varepsilon}(\tilde{\tau}f_j)f_T\tilde{\nabla}_{\tilde{\tau}f_j}^{\mathcal{F},\beta,\varepsilon}(\tau\sigma) \right\rangle kdv_{\mathcal{M}_x} \\
&= \left\langle c_{\beta,\varepsilon}(\tilde{\tau}f_i)(1-Q)\tilde{\nabla}_{\tilde{\tau}f_i}^{\mathcal{F},\beta,\varepsilon}(\tau\sigma), c_{\beta,\varepsilon}(\tilde{\tau}f_j)(1-Q)\tilde{\nabla}_{\tilde{\tau}f_j}^{\mathcal{F},\beta,\varepsilon}(\tau\sigma) \right\rangle_x \\
&\quad + O\left(\frac{1}{\sqrt{T}}\right)|\sigma|_x^2 + O\left(\frac{1}{\sqrt{T}}\right)\sum_{j=1}^{q+q_1} \left| {}^Q\tilde{\nabla}_{\tilde{\tau}f_j}^{\mathcal{F},\beta,\varepsilon}(\tau\sigma) \right|_x^2.
\end{aligned}$$

By definition (cf. (1.66)), one has on  $s(M)$  that

$$\begin{aligned}
(2.57) \quad & (1-Q)\left(\tilde{\nabla}_{f_i}^{\mathcal{F},\beta,\varepsilon}\right)Q = \frac{\beta}{2}\sum_{k=1}^q\sum_{j=1}^{q_2}\left\langle \nabla_{f_i}^{T\mathcal{M},\beta,\varepsilon}e_j, f_k \right\rangle c_{\beta,\varepsilon}(e_j)c_{\beta,\varepsilon}(\beta^{-1}f_k) \\
&\quad + \frac{\varepsilon^{-1}}{2}\sum_{k=q+1}^{q+q_1}\sum_{j=1}^{q_2}\left\langle \nabla_{f_i}^{T\mathcal{M},\beta,\varepsilon}e_j, f_k \right\rangle c_{\beta,\varepsilon}(e_j)c_{\beta,\varepsilon}(\varepsilon f_k).
\end{aligned}$$

By (2.41), one has for  $1 \leq i, k \leq q$  that

$$(2.58) \quad \left\langle \nabla_{f_i}^{T\mathcal{M},\beta,\varepsilon}e_j, f_k \right\rangle = 0.$$

Also, by (1.5) and (1.8), one finds that when  $1 \leq i \leq q, q+1 \leq k \leq q+q_1$ ,

$$(2.59) \quad \varepsilon^{-1}\left\langle \nabla_{f_i}^{T\mathcal{M},\beta,\varepsilon}e_j, f_k \right\rangle = O(\varepsilon).$$

From (2.56)-(2.59), one gets that if  $1 \leq i, j \leq q$  with  $i \neq j$ , then

$$\begin{aligned}
(2.60) \quad & \int_{\mathcal{M}_x} \left\langle (1-p_{T,\beta,\varepsilon})c_{\beta,\varepsilon}(\tilde{\tau}f_i)f_T\tilde{\nabla}_{\tilde{\tau}f_i}^{\mathcal{F},\beta,\varepsilon}(\tau\sigma), (1-p_{T,\beta,\varepsilon})c_{\beta,\varepsilon}(\tilde{\tau}f_j)f_T\tilde{\nabla}_{\tilde{\tau}f_j}^{\mathcal{F},\beta,\varepsilon}(\tau\sigma) \right\rangle kdv_{\mathcal{M}_x} \\
&= O\left(\frac{\varepsilon^2}{\beta^2} + \frac{1}{\sqrt{T}}\right)|\sigma|_x^2 + O\left(\frac{1}{\sqrt{T}}\right)\sum_{j=1}^{q+q_1} \left| {}^Q\tilde{\nabla}_{\tilde{\tau}f_j}^{\mathcal{F},\beta,\varepsilon}(\tau\sigma) \right|_x^2.
\end{aligned}$$

If  $q+1 \leq i \leq q+q_1$ ,  $1 \leq k \leq q$ , then one has

$$(2.61) \quad \beta \left\langle \nabla_{f_i}^{T\mathcal{M}, \beta, \varepsilon} e_j, f_k \right\rangle = O\left(\frac{1}{\beta}\right),$$

while if  $q+1 \leq i$ ,  $k \leq q+q_1$ , one has

$$(2.62) \quad \varepsilon^{-1} \left\langle \nabla_{f_i}^{T\mathcal{M}, \beta, \varepsilon} e_j, f_k \right\rangle = O(\varepsilon^{-1}).$$

Combining with (2.56)-(2.59), one gets that if  $q+1 \leq i \leq q+q_1$ ,  $1 \leq j \leq q$ , then

$$(2.63) \quad \begin{aligned} & \int_{\mathcal{M}_x} \left\langle (1 - p_{T, \beta, \varepsilon}) c_{\beta, \varepsilon} (\tilde{\tau} f_i) f_T \tilde{\nabla}_{\tilde{\tau} f_i}^{\mathcal{F}, \beta, \varepsilon} (\tau \sigma), (1 - p_{T, \beta, \varepsilon}) c_{\beta, \varepsilon} (\tilde{\tau} f_j) f_T \tilde{\nabla}_{\tilde{\tau} f_j}^{\mathcal{F}, \beta, \varepsilon} (\tau \sigma) \right\rangle k dv_{\mathcal{M}_x} \\ &= O\left(\frac{\varepsilon(\beta + \varepsilon)}{\beta^2} + \frac{1}{\sqrt{T}}\right) |\sigma|_x^2 + O\left(\frac{1}{\sqrt{T}}\right) \sum_{j=1}^{q+q_1} \left| Q \tilde{\nabla}_{f_j}^{\mathcal{F}, \beta, \varepsilon} (\tau \sigma) \right|_x^2. \end{aligned}$$

Also, when  $q+1 \leq i, j \leq q+q_1$  with  $i \neq j$ , one gets

$$(2.64) \quad \begin{aligned} & \int_{\mathcal{M}_x} \left\langle (1 - p_{T, \beta, \varepsilon}) c_{\beta, \varepsilon} (\tilde{\tau} f_i) f_T \tilde{\nabla}_{\tilde{\tau} f_i}^{\mathcal{F}, \beta, \varepsilon} (\tau \sigma), (1 - p_{T, \beta, \varepsilon}) c_{\beta, \varepsilon} (\tilde{\tau} f_j) f_T \tilde{\nabla}_{\tilde{\tau} f_j}^{\mathcal{F}, \beta, \varepsilon} (\tau \sigma) \right\rangle k dv_{\mathcal{M}_x} \\ &= O\left(\frac{(\beta + \varepsilon)^2}{\beta^2} + \frac{1}{\sqrt{T}}\right) |\sigma|_x^2 + O\left(\frac{1}{\sqrt{T}}\right) \sum_{j=1}^{q+q_1} \left| Q \tilde{\nabla}_{f_j}^{\mathcal{F}, \beta, \varepsilon} (\tau \sigma) \right|_x^2. \end{aligned}$$

Now we consider the term which corresponds to what in (2.47) but with  $f_j$ ,  $1 \leq j \leq q+q_1$ , being replaced by  $e_k$ ,  $1 \leq k \leq q_2$ . That is, we consider the term

$$(2.65) \quad \begin{aligned} & \left\langle (1 - p_{T, \beta, \varepsilon}) c_{\beta, \varepsilon} (\tilde{\tau} f_i) \tilde{\nabla}_{\tilde{\tau} f_i}^{\mathcal{F}, \beta, \varepsilon} J_{T, \beta, \varepsilon} \sigma, (1 - p_{T, \beta, \varepsilon}) c_{\beta, \varepsilon} (\tau e_k) \tilde{\nabla}_{\tau e_k}^{\mathcal{F}, \beta, \varepsilon} J_{T, \beta, \varepsilon} \sigma \right\rangle \\ &= \langle c_{\beta, \varepsilon} (\tilde{\tau} f_i) (1 - p_{T, \beta, \varepsilon}) \tilde{\tau} f_i (f_T) \tau \sigma, c_{\beta, \varepsilon} (\tau e_k) \tau e_k (f_T) \tau \sigma \rangle \\ &+ \left\langle (1 - p_{T, \beta, \varepsilon}) c_{\beta, \varepsilon} (\tilde{\tau} f_i) f_T \nabla_{\tilde{\tau} f_i}^{\mathcal{F}, \beta, \varepsilon} \tau \sigma, c_{\beta, \varepsilon} (\tau e_k) f_T \tilde{\nabla}_{\tau e_k}^{\mathcal{F}, \beta, \varepsilon} (\tau \sigma) \right\rangle \\ &+ \left\langle (1 - p_{T, \beta, \varepsilon}) c_{\beta, \varepsilon} (\tilde{\tau} f_i) f_T \tilde{\nabla}_{\tilde{\tau} f_i}^{\mathcal{F}, \beta, \varepsilon} (\tau \sigma), c_{\beta, \varepsilon} (\tau e_k) \tau e_k (f_T) \tau \sigma \right\rangle \\ &+ \left\langle c_{\beta, \varepsilon} (\tilde{\tau} f_i) (1 - p_{T, \beta, \varepsilon}) \tilde{\tau} f_i (f_T) \tau \sigma, c_{\beta, \varepsilon} (\tau e_k) f_T \tilde{\nabla}_{\tau e_k}^{\mathcal{F}, \beta, \varepsilon} (\tau \sigma) \right\rangle. \end{aligned}$$

First, by (2.48) and the obvious parity consideration, we have

$$(2.66) \quad \langle c_{\beta, \varepsilon} (\tilde{\tau} f_i) (1 - p_{T, \beta, \varepsilon}) \tilde{\tau} f_i (f_T) \tau \sigma, c_{\beta, \varepsilon} (\tau e_k) \tau e_k (f_T) \tau \sigma \rangle = 0.$$

**Lemma 2.8.** *For any  $U \in \Gamma(\mathcal{F}_2^\perp|_{s(M)})$ , the following identity holds on  $s(M)$ ,*

$$(2.67) \quad \left( Q \tilde{\nabla}_U^{\mathcal{F}, \beta, \varepsilon} (\tau \sigma) \right) \Big|_{s(M)} = 0.$$

*Proof.* By construction, one has

$$(2.68) \quad Q \nabla_Z^{\mathcal{F}, \phi_1(\mathcal{F}_1^\perp), \beta, \varepsilon} (\tau \sigma) = 0.$$

Taking derivative with respect to  $z_i$ , one gets

$$(2.69) \quad \left( Q \nabla_{e_i}^{\mathcal{F}, \phi_1(\mathcal{F}_1^\perp), \beta, \varepsilon}(\tau\sigma) \right) \Big|_{s(M)} = 0.$$

Formula (2.67) follows from (2.69).  $\square$

For the second term in the right hand side of (2.65), for any  $x \in s(M)$ , by (2.26), (2.44) and Lemma 2.8, one has

$$(2.70) \quad \int_{\mathcal{M}_x} \left\langle (1 - p_{T, \beta, \varepsilon}) c_{\beta, \varepsilon}(\tilde{\tau} f_i) f_T \tilde{\nabla}_{\tilde{\tau} f_i}^{\mathcal{F}, \beta, \varepsilon}(\tau\sigma), c_{\beta, \varepsilon}(\tau e_k) f_T \tilde{\nabla}_{\tau e_k}^{\mathcal{F}, \beta, \varepsilon}(\tau\sigma) \right\rangle_{(x, Z)} k dv_{\mathcal{M}_x} \\ = \left\langle c_{\beta, \varepsilon}(\tilde{\tau} f_i) (1 - Q) \tilde{\nabla}_{\tilde{\tau} f_i}^{\mathcal{F}, \beta, \varepsilon}(\tau\sigma), c_{\beta, \varepsilon}(e_k) (1 - Q) \tilde{\nabla}_{e_k}^{\mathcal{F}, \beta, \varepsilon}(\tau\sigma) \right\rangle_x \\ + O\left(\frac{1}{\sqrt{T}}\right) |\sigma|_x^2 + O\left(\frac{1}{\sqrt{T}}\right) \sum_{j=1}^{q+q_1} \left| Q \tilde{\nabla}_{f_j}^{\mathcal{F}, \beta, \varepsilon}(\tau\sigma) \right|_x^2.$$

By (1.6) and (2.1), one knows that for any  $U, V \in \Gamma(\mathcal{F}_2^\perp)$  and  $X \in \Gamma(\mathcal{F})$ , one has

$$(2.71) \quad \left\langle \nabla_U^{T\mathcal{M}, \beta, \varepsilon} V, X \right\rangle = 0.$$

Similarly as in (2.57), one has by (2.71) that, on  $s(M)$ ,

$$(2.72) \quad (1 - Q) \left( \tilde{\nabla}_{e_k}^{\mathcal{F}, \beta, \varepsilon} \right) Q = \frac{\beta}{2} \sum_{s=1}^q \sum_{j=1}^{q_2} \left\langle \nabla_{e_k}^{T\mathcal{M}, \beta, \varepsilon} e_j, f_s \right\rangle c_{\beta, \varepsilon}(e_j) c_{\beta, \varepsilon}(\beta^{-1} f_s) \\ + \frac{\varepsilon^{-1}}{2} \sum_{s=q+1}^{q+q_1} \sum_{j=1}^{q_2} \left\langle \nabla_{e_k}^{T\mathcal{M}, \beta, \varepsilon} e_j, f_s \right\rangle c_{\beta, \varepsilon}(e_j) c_{\beta, \varepsilon}(\varepsilon f_s) \\ = \frac{\varepsilon^{-1}}{2} \sum_{s=q+1}^{q+q_1} \sum_{j=1}^{q_2} \left\langle \nabla_{e_k}^{T\mathcal{M}, \beta, \varepsilon} e_j, f_s \right\rangle c_{\beta, \varepsilon}(e_j) c_{\beta, \varepsilon}(\varepsilon f_s).$$

From (2.57), (2.70), (2.72) and the easy parity consideration, one gets that for  $1 \leq i \leq q + q_1$ ,  $1 \leq k \leq q_2$ ,

$$(2.73) \quad \int_{\mathcal{M}_x} \left\langle (1 - p_{T, \beta, \varepsilon}) c_{\beta, \varepsilon}(\tilde{\tau} f_i) f_T \tilde{\nabla}_{\tilde{\tau} f_i}^{\mathcal{F}, \beta, \varepsilon}(\tau\sigma), c_{\beta, \varepsilon}(\tau e_k) f_T \tilde{\nabla}_{\tau e_k}^{\mathcal{F}, \beta, \varepsilon}(\tau\sigma) \right\rangle_{(x, Z)} k dv_{\mathcal{M}_x} \\ = O\left(\frac{1}{\sqrt{T}}\right) |\sigma|_x^2 + O\left(\frac{1}{\sqrt{T}}\right) \sum_{j=1}^{q+q_1} \left| Q \nabla_{f_j}^{\mathcal{F}, \phi_1(\mathcal{F}_1^\perp), \beta, \varepsilon}(\tau\sigma) \right|_x^2.$$

For the third term in the right hand side of (2.65), if  $1 \leq i \leq q + q_1$ , one has by an easy degree consideration,

$$(2.74) \quad \left\langle (1 - p_{T, \beta, \varepsilon}) c_{\beta, \varepsilon}(\tilde{\tau} f_i) f_T \tilde{\nabla}_{\tilde{\tau} f_i}^{\mathcal{F}, \beta, \varepsilon}(\tau\sigma), c_{\beta, \varepsilon}(\tau e_k) \tau e_k(f_T) \tau\sigma \right\rangle \\ = \left\langle c_{\beta, \varepsilon}(\tilde{\tau} f_i) f_T \tilde{\nabla}_{\tilde{\tau} f_i}^{\mathcal{F}, \beta, \varepsilon}(\tau\sigma), c_{\beta, \varepsilon}(\tau e_k) \tau e_k(f_T) \tau\sigma \right\rangle \\ = \left\langle c_{\beta, \varepsilon}(\tilde{\tau} f_i) f_T (1 - Q) \tilde{\nabla}_{\tilde{\tau} f_i}^{\mathcal{F}, \beta, \varepsilon}(\tau\sigma), c_{\beta, \varepsilon}(\tau e_k) \tau e_k(f_T) \tau\sigma \right\rangle.$$

As in (2.57), one has

$$(2.75) \quad (1 - Q) \left( \tilde{\nabla}_{\tau f_i}^{\mathcal{F}, \beta, \varepsilon} \right) Q = \frac{1}{2\beta} \sum_{k=1}^q \sum_{j=1}^{q_2} \left\langle \nabla_{\tau f_i}^{T\mathcal{M}, \beta, \varepsilon} (\tau e_j), \tau f_k \right\rangle_{\beta, \varepsilon} c_{\beta, \varepsilon} (\tau e_j) c_{\beta, \varepsilon} (\beta^{-1} \tau f_k) \\ + \frac{\varepsilon}{2} \sum_{k=q+1}^{q+q_1} \sum_{j=1}^{q_2} \left\langle \nabla_{\tau f_i}^{T\mathcal{M}, \beta, \varepsilon} (\tau e_j), \tau f_k \right\rangle_{\beta, \varepsilon} c_{\beta, \varepsilon} (\tau e_j) c_{\beta, \varepsilon} (\varepsilon \tau f_k),$$

where the subscripts “ $\beta$ ”, “ $\varepsilon$ ” are to emphasize that the pointwise inner product is the one with respect to  $g_{\beta, \varepsilon}^{T\mathcal{M}}$ .

From (2.75), one finds

$$(2.76) \quad \left\langle c_{\beta, \varepsilon} (\tau f_i) f_T (1 - Q) \tilde{\nabla}_{\tau f_i}^{\mathcal{F}, \beta, \varepsilon} (\tau \sigma), c_{\beta, \varepsilon} (\tau e_k) \tau e_k (f_T) \tau \sigma \right\rangle \\ = \frac{1}{2\beta} \sum_{m=1}^q \sum_{j=1}^{q_2} \left( \int_{s(M)} \left\langle c_{\beta, \varepsilon} (f_i) c_{\beta, \varepsilon} (e_j) c_{\beta, \varepsilon} (\beta^{-1} f_m) \sigma, c_{\beta, \varepsilon} (e_k) \sigma \right\rangle dv_{s(M)} \right. \\ \left. \cdot \int_{\mathcal{M}_x} \left\langle \nabla_{\tau f_i}^{T\mathcal{M}, \beta, \varepsilon} (\tau e_j), \tau f_m \right\rangle_{\beta, \varepsilon} f_T \tau e_k (f_T) k dv_{\mathcal{M}_x}(Z) \right) \\ + \frac{\varepsilon}{2} \sum_{m=q+1}^{q+q_1} \sum_{j=1}^{q_2} \left( \int_{s(M)} \left\langle c_{\beta, \varepsilon} (f_i) c_{\beta, \varepsilon} (e_j) c_{\beta, \varepsilon} (\varepsilon f_m) \sigma, c_{\beta, \varepsilon} (e_k) \sigma \right\rangle dv_{s(M)} \right. \\ \left. \cdot \int_{\mathcal{M}_x} \left\langle \nabla_{\tau f_i}^{T\mathcal{M}, \beta, \varepsilon} (\tau e_j), \tau f_m \right\rangle_{\beta, \varepsilon} f_T \tau e_k (f_T) k dv_{\mathcal{M}_x}(Z) \right) \\ = -\frac{1}{2\beta} \sum_{m=1}^q \int_{s(M)} \left\langle c_{\beta, \varepsilon} (f_i) c_{\beta, \varepsilon} (\beta^{-1} f_m) \sigma, \sigma \right\rangle dv_{s(M)} \int_{\mathcal{M}_x} \left\langle \nabla_{\tau f_i}^{T\mathcal{M}, \beta, \varepsilon} (\tau e_k), \tau f_m \right\rangle_{\beta, \varepsilon} f_T \tau e_k (f_T) k dv_{\mathcal{M}_x}(Z) \\ - \frac{\varepsilon}{2} \sum_{m=q+1}^{q+q_1} \int_{s(M)} \left\langle c_{\beta, \varepsilon} (f_i) c_{\beta, \varepsilon} (\varepsilon f_m) \sigma, \sigma \right\rangle dv_{s(M)} \int_{\mathcal{M}_x} \left\langle \nabla_{\tau f_i}^{T\mathcal{M}, \beta, \varepsilon} (\tau e_k), \tau f_m \right\rangle_{\beta, \varepsilon} f_T \tau e_k (f_T) k dv_{\mathcal{M}_x}(Z).$$

Clearly, when  $i \neq m$ ,  $c(f_i)c(f_m)$  is skew-adjoint, thus

$$(2.77) \quad \left\langle c_{\beta, \varepsilon} (f_i) c_{\beta, \varepsilon} (f_m) \sigma, \sigma \right\rangle = 0.$$

By (2.23), one has

$$(2.78) \quad \tau e_k (f_T) (x, Z) = \left( -\frac{\tau e_k(k)\gamma}{2k^{3/2}\sqrt{\alpha_T}} + \frac{\tau e_k(\gamma)}{k^{1/2}\sqrt{\alpha_T}} - \frac{T\tau e_k(|Z|^2)\gamma}{2k^{1/2}\sqrt{\alpha_T}} \right) \exp \left( -\frac{T|Z|^2}{2} \right).$$

By (2.3), one knows that  $\tau e_k$  does not depend on  $\beta$  and  $\varepsilon$ .

From Lemma 2.6, one gets that for  $1 \leq i, m \leq q, 1 \leq j \leq q_2$ ,

$$(2.79) \quad \left\langle \nabla_{\tau f_i}^{T\mathcal{M}, \beta, \varepsilon} (\tau e_j), \tau f_m \right\rangle_{\beta, \varepsilon} \Big|_{(x, Z)} = \left\langle \nabla_{f'_i + \sum_{k=q+1}^{q+q_1} O(\varepsilon^2|Z|)f'_k}^{T\mathcal{M}, \beta, \varepsilon} (\tau e_j), f'_m + \sum_{k=q+1}^{q+q_1} O(\varepsilon^2|Z|)f'_k \right\rangle_{\beta, \varepsilon} \\ + O(|Z|^2) = O(\varepsilon^2|Z|) + O(|Z|^2).$$

From (2.78) and (2.79), one gets

$$(2.80) \quad \frac{1}{\beta} \int_{\mathcal{M}_x} \left\langle \nabla_{\tau f_i}^{TM, \beta, \varepsilon}(\tau e_j), \tau f_m \right\rangle_{\beta, \varepsilon} f_T \tau e_k(f_T) k dv_{\mathcal{M}_x}(Z) = O\left(\frac{\varepsilon^2}{\beta} + \frac{1}{\sqrt{T}}\right).$$

From (2.74), (2.76), (2.77) and (2.80), one finds that when  $1 \leq i \leq q$ ,  $1 \leq k \leq q_2$ ,

$$(2.81) \quad \int_{\mathcal{M}_x} \left\langle (1 - p_{T, \beta, \varepsilon}) c_{\beta, \varepsilon}(\tilde{\tau} f_i) f_T \tilde{\nabla}_{\tilde{\tau} f_i}^{\mathcal{F}, \beta, \varepsilon}(\tau \sigma), c_{\beta, \varepsilon}(\tau e_k) \tau e_k(f_T) \tau \sigma \right\rangle_{(x, Z)} k(x, Z) dv_{\mathcal{M}_x}(Z) \\ = O\left(\frac{\varepsilon^2}{\beta^2} + \frac{1}{\sqrt{T}}\right) |\sigma|_x^2.$$

Now for  $q+1 \leq i$ ,  $m \leq q+q_1$  and  $1 \leq j \leq q_2$ , one has

$$(2.82) \quad \left\langle \nabla_{\tau f_i}^{TM, \beta, \varepsilon}(\tau e_j), \tau f_m \right\rangle_{\beta, \varepsilon} \Big|_{(x, Z)} = \left\langle \nabla_{f'_i + \sum_{j=1}^q O\left(\frac{|Z|}{\beta^2}\right) f'_j + \sum_{k=q+1}^{q+q_1} O(|Z|) f'_k}^{TM, \beta, \varepsilon}(\tau e_j), \right. \\ \left. f'_m + \sum_{j=1}^q O\left(\frac{|Z|}{\beta^2}\right) f'_j + \sum_{k=q+1}^{q+q_1} O(|Z|) f'_k \right\rangle_{\beta, \varepsilon} + O(|Z|^2) \\ = O\left(\frac{1}{\varepsilon^2}\right) + O\left(\left(\frac{1}{\beta^2} + \frac{1}{\varepsilon^2}\right) |Z|\right) + O(|Z|^2).$$

By using (2.74), (2.76)-(2.78) and (2.82), one finds that when  $q+1 \leq i \leq q+q_1$ ,  $1 \leq k \leq q_2$ ,

$$(2.83) \quad \int_{\mathcal{M}_x} \left\langle (1 - p_{T, \beta, \varepsilon}) c_{\beta, \varepsilon}(\tilde{\tau} f_i) f_T \tilde{\nabla}_{\tilde{\tau} f_i}^{\mathcal{F}, \beta, \varepsilon} \tau \sigma, c_{\beta, \varepsilon}(\tau e_k) \tau e_k(f_T) \tau \sigma \right\rangle_{(x, Z)} k(x, Z) dv_{\mathcal{M}_x}(Z) \\ = O\left(1 + \frac{\varepsilon^2}{\beta^2} + \frac{1}{\sqrt{T}}\right) |\sigma|_x^2.$$

For the fourth term in the right hand side of (2.65), one verifies easily that

$$(2.84) \quad \left\langle c_{\beta, \varepsilon}(\tau f_i) (1 - p_{T, \beta, \varepsilon}) \tau f_i(f_T) \tau \sigma, c_{\beta, \varepsilon}(\tau e_k) f_T \tilde{\nabla}_{\tau e_k}^{\mathcal{F}, \beta, \varepsilon}(\tau \sigma) \right\rangle \\ = \left\langle c_{\beta, \varepsilon}(\tau f_i) (1 - p_{T, \beta, \varepsilon}) \tau f_i(f_T) \tau \sigma, c_{\beta, \varepsilon}(\tau e_k) f_T (1 - Q) \tilde{\nabla}_{\tau e_k}^{\mathcal{F}, \beta, \varepsilon}(\tau \sigma) \right\rangle \\ = \left\langle c_{\beta, \varepsilon}(\tau f_i) \rho_{T, \beta, \varepsilon, i} \tau \sigma, c_{\beta, \varepsilon}(\tau e_k) f_T (1 - Q) \tilde{\nabla}_{\tau e_k}^{\mathcal{F}, \beta, \varepsilon}(\tau \sigma) \right\rangle.$$

As in (2.75), one has

$$(2.85) \quad (1 - Q) \tilde{\nabla}_{\tau e_k}^{\mathcal{F}, \beta, \varepsilon}(\tau \sigma) = \frac{1}{2\beta} \sum_{j=1}^{q_2} \sum_{m=1}^q \left\langle \nabla_{\tau e_k}^{TM, \beta, \varepsilon}(\tau e_j), \tau f_m \right\rangle_{\beta, \varepsilon} c_{\beta, \varepsilon}(\tau e_j) c_{\beta, \varepsilon}(\beta^{-1} \tau f_m) \tau \sigma \\ + \frac{\varepsilon}{2} \sum_{j=1}^{q_2} \sum_{m=q+1}^{q+q_1} \left\langle \nabla_{\tau e_k}^{TM, \beta, \varepsilon}(\tau e_j), \tau f_m \right\rangle_{\beta, \varepsilon} c_{\beta, \varepsilon}(\tau e_j) c_{\beta, \varepsilon}(\varepsilon \tau f_m) \tau \sigma.$$



By Lemma 2.6, (2.1) and (2.71), one verifies that for  $1 \leq m \leq q$ , one has

$$(2.86) \quad \left\langle \nabla_{\tau e_i}^{TM, \beta, \varepsilon}(\tau e_j), \tau f_m \right\rangle_{\beta, \varepsilon} \Big|_{(x, Z)} = \left\langle \nabla_{\tau e_i}^{TM, \beta, \varepsilon} \tau e_j, f'_m + \sum_{k=q+1}^{q+q_1} O(\varepsilon^2 |Z|) f'_k \right\rangle_{\beta, \varepsilon} + O(|Z|^2) \\ = O(\varepsilon^2 |Z|) + O(|Z|^2),$$

while for  $q+1 \leq m \leq q+q_1$ , one has,

$$(2.87) \quad \left\langle \nabla_{\tau e_i}^{TM, \beta, \varepsilon}(\tau e_j), \tau f_m \right\rangle_{\beta, \varepsilon} \Big|_{(x, Z)} = \left\langle \nabla_{\tau e_i}^{TM, \beta, \varepsilon} \tau e_j, f'_m + \sum_{j=1}^q O\left(\frac{|Z|}{\beta^2}\right) f'_j + \sum_{k=q+1}^{q+q_1} O(|Z|) f'_k \right\rangle_{\beta, \varepsilon} \\ + O(|Z|^2) = O(1) + O(|Z|) + O(|Z|^2).$$

From (2.52), (2.53) and (2.84)-(2.87), one gets that for  $1 \leq i \leq q$  and  $1 \leq k \leq q_2$ , and also using the parity consideration,

$$(2.88) \quad \frac{1}{\beta} \left\langle c_{\beta, \varepsilon}(\beta^{-1} \tau f_i) (1 - p_{T, \beta, \varepsilon}) \tau f_i (f_T) \tau \sigma, c_{\beta, \varepsilon}(\tau e_k) f_T \tilde{\nabla}_{\tau e_k}^{\mathcal{F}, \beta, \varepsilon}(\tau \sigma) \right\rangle \\ = O\left(\frac{\varepsilon^2}{\beta^2} + \frac{1}{\sqrt{T}}\right) \int_{s(M)} |\sigma|_x^2 dv_{s(M)},$$

while for  $q+1 \leq i \leq q+q_1$  and  $1 \leq k \leq q_2$ , one has

$$(2.89) \quad \varepsilon \left\langle c_{\beta, \varepsilon}(\varepsilon \tau f_i) (1 - p_{T, \beta, \varepsilon}) \tau f_i (f_T) \tau \sigma, c_{\beta, \varepsilon}(\tau e_k) f_T \tilde{\nabla}_{\tau e_k}^{\mathcal{F}, \beta, \varepsilon}(\tau \sigma) \right\rangle \\ = O\left(\varepsilon^2 + \frac{1}{\sqrt{T}}\right) \int_{s(M)} |\sigma|_x^2 dv_{s(M)}.$$

Now we consider the term for  $1 \leq i, k \leq q_2$  with  $i \neq k$ ,

$$(2.90) \quad \left\langle (1 - p_{T, \beta, \varepsilon}) c_{\beta, \varepsilon}(\tau e_i) \tilde{\nabla}_{\tau e_i}^{\mathcal{F}, \beta, \varepsilon} J_{T, \beta, \varepsilon} \sigma, c_{\beta, \varepsilon}(\tau e_k) \tilde{\nabla}_{\tau e_k}^{\mathcal{F}, \beta, \varepsilon} J_{T, \beta, \varepsilon} \sigma \right\rangle \\ = \left\langle (1 - p_{T, \beta, \varepsilon}) c_{\beta, \varepsilon}(\tau e_i) \tau e_i (f_T) \tau \sigma, c_{\beta, \varepsilon}(\tau e_k) \tau e_k (f_T) \tau \sigma \right\rangle \\ + \left\langle (1 - p_{T, \beta, \varepsilon}) c_{\beta, \varepsilon}(\tau e_i) f_T \tilde{\nabla}_{\tau e_i}^{\mathcal{F}, \beta, \varepsilon}(\tau \sigma), c_{\beta, \varepsilon}(\tau e_k) f_T \tilde{\nabla}_{\tau e_k}^{\mathcal{F}, \beta, \varepsilon}(\tau \sigma) \right\rangle \\ + \left\langle (1 - p_{T, \beta, \varepsilon}) c_{\beta, \varepsilon}(\tau e_i) f_T \tilde{\nabla}_{\tau e_i}^{\mathcal{F}, \beta, \varepsilon}(\tau \sigma), c_{\beta, \varepsilon}(\tau e_k) \tau e_k (f_T) \tau \sigma \right\rangle \\ + \left\langle (1 - p_{T, \beta, \varepsilon}) c_{\beta, \varepsilon}(\tau e_i) \tau e_i (f_T) \tau \sigma, c_{\beta, \varepsilon}(\tau e_k) f_T \tilde{\nabla}_{\tau e_k}^{\mathcal{F}, \beta, \varepsilon}(\tau \sigma) \right\rangle.$$

For the first term in the right hand side of (2.90), one has, as  $i \neq k$ ,

$$(2.91) \quad \left\langle (1 - p_{T, \beta, \varepsilon}) c_{\beta, \varepsilon}(\tau e_i) \tau e_i (f_T) \tau \sigma, c_{\beta, \varepsilon}(\tau e_k) \tau e_k (f_T) \tau \sigma \right\rangle \\ = - \left\langle \tau e_k (f_T) \tau e_i (f_T) \tau \sigma, c_{\beta, \varepsilon}(\tau e_i) c_{\beta, \varepsilon}(\tau e_k) \tau \sigma \right\rangle = 0.$$

For the second term in the right hand side of (2.90), one has by (2.26) and Lemma 2.8 that for any  $x \in s(M)$ ,

$$\begin{aligned}
(2.92) \quad & \int_{\mathcal{M}_x} \left\langle (1 - p_{T,\beta,\varepsilon}) c_{\beta,\varepsilon}(\tau e_i) f_T \widetilde{\nabla}_{\tau e_i}^{\mathcal{F},\beta,\varepsilon}(\tau\sigma), c_{\beta,\varepsilon}(\tau e_k) f_T \widetilde{\nabla}_{\tau e_k}^{\mathcal{F},\beta,\varepsilon}(\tau\sigma) \right\rangle_{(x,Z)} kdv_{\mathcal{M}_x} \\
&= \int_{\mathcal{M}_x} f_T^2 \left\langle (1 - Q) c_{\beta,\varepsilon}(\tau e_i) (1 - Q) \widetilde{\nabla}_{\tau e_i}^{\mathcal{F},\beta,\varepsilon}(\tau\sigma), c_{\beta,\varepsilon}(\tau e_k) (1 - Q) \widetilde{\nabla}_{\tau e_k}^{\mathcal{F},\beta,\varepsilon}(\tau\sigma) \right\rangle_{(x,Z)} kdv_{\mathcal{M}_x} \\
&\quad + O\left(\frac{1}{\sqrt{T}}\right) |\sigma|_x^2 + O\left(\frac{1}{\sqrt{T}}\right) \sum_{i=1}^{q+q_1} \left| Q \nabla_{f_i}^{\mathcal{F},\phi_1(\mathcal{F}_1^\perp),\beta,\varepsilon}(\tau\sigma) \right|_x^2 \\
&= \left\langle (1 - Q) c_{\beta,\varepsilon}(e_i) (1 - Q) \widetilde{\nabla}_{e_i}^{\mathcal{F},\beta,\varepsilon}(\tau\sigma), c_{\beta,\varepsilon}(e_k) (1 - Q) \widetilde{\nabla}_{e_k}^{\mathcal{F},\beta,\varepsilon}(\tau\sigma) \right\rangle_x \\
&\quad + O\left(\frac{1}{\sqrt{T}}\right) |\sigma|_x^2 + O\left(\frac{1}{\sqrt{T}}\right) \sum_{i=1}^{q+q_1} \left| Q \nabla_{f_i}^{\mathcal{F},\phi_1(\mathcal{F}_1^\perp),\beta,\varepsilon}(\tau\sigma) \right|_x^2.
\end{aligned}$$

Now, one has by (2.71) that for any  $1 \leq i \leq q_2$ , at  $x \in s(M)$ ,

$$\begin{aligned}
(2.93) \quad & (1-Q)c_{\beta,\varepsilon}(e_i)(1-Q)\widetilde{\nabla}_{e_i}^{\mathcal{F},\beta,\varepsilon}Q = \frac{\beta}{2} \sum_{j=1, j \neq i}^{q_2} \sum_{m=1}^q \langle \nabla_{e_i}^{TM,\beta,\varepsilon} e_j, f_m \rangle c_{\beta,\varepsilon}(e_i) c_{\beta,\varepsilon}(e_j) c_{\beta,\varepsilon}(\beta^{-1} f_m) \\
&\quad + \frac{\varepsilon^{-1}}{2} \sum_{j=1, j \neq i}^{q_2} \sum_{m=q+1}^{q+q_1} \langle \nabla_{e_i}^{TM,\beta,\varepsilon} e_j, f_m \rangle c_{\beta,\varepsilon}(e_i) c_{\beta,\varepsilon}(e_j) c_{\beta,\varepsilon}(\varepsilon f_m) \\
&= \frac{\varepsilon^{-1}}{2} \sum_{j=1, j \neq i}^{q_2} \sum_{m=q+1}^{q+q_1} \langle \nabla_{e_i}^{TM,\beta,\varepsilon} e_j, f_m \rangle c_{\beta,\varepsilon}(e_i) c_{\beta,\varepsilon}(e_j) c_{\beta,\varepsilon}(\varepsilon f_m).
\end{aligned}$$

For  $q+1 \leq m \leq q+q_1$ , one has, by (2.1),

$$(2.94) \quad \langle \nabla_{e_i}^{TM,\beta,\varepsilon} e_j, f_m \rangle = O(\varepsilon^2).$$

From (2.92)-(2.94), one gets that for  $x \in s(M)$ ,

$$\begin{aligned}
(2.95) \quad & \int_{\mathcal{M}_x} \left\langle (1 - p_{T,\beta,\varepsilon}) c_{\beta,\varepsilon}(\tau e_i) f_T \widetilde{\nabla}_{\tau e_i}^{\mathcal{F},\beta,\varepsilon}(\tau\sigma), c_{\beta,\varepsilon}(\tau e_k) f_T \widetilde{\nabla}_{\tau e_k}^{\mathcal{F},\beta,\varepsilon}(\tau\sigma) \right\rangle_{(x,Z)} kdv_{\mathcal{M}_x} \\
&= O\left(\varepsilon^2 + \frac{1}{\sqrt{T}}\right) |\sigma|_x^2 + O\left(\frac{1}{\sqrt{T}}\right) \sum_{i=1}^{q+q_1} \left| Q \nabla_{f_i}^{\mathcal{F},\phi_1(\mathcal{F}_1^\perp),\beta,\varepsilon}(\tau\sigma) \right|_x^2.
\end{aligned}$$

For the third term in the right hand side of (2.90), since  $i \neq k$ , by (2.85) and a simple parity consideration, one has that

$$\begin{aligned}
(2.96) \quad & \left\langle (1 - p_{T,\beta,\varepsilon}) c_{\beta,\varepsilon}(\tau e_i) f_T \widetilde{\nabla}_{\tau e_i}^{\mathcal{F},\beta,\varepsilon}(\tau\sigma), c_{\beta,\varepsilon}(\tau e_k) \tau e_k(f_T)(\tau\sigma) \right\rangle \\
&= \left\langle c_{\beta,\varepsilon}(\tau e_i) f_T \widetilde{\nabla}_{\tau e_i}^{\mathcal{F},\beta,\varepsilon}(\tau\sigma), c_{\beta,\varepsilon}(\tau e_k) \tau e_k(f_T) \tau\sigma \right\rangle = 0.
\end{aligned}$$

Similarly, for the fourth term in the right hand side of (2.90), one has

$$(2.97) \quad \left\langle (1 - p_{T,\beta,\varepsilon}) c_{\beta,\varepsilon}(\tau e_i) \tau e_i(f_T) \tau\sigma, c_{\beta,\varepsilon}(\tau e_k) f_T \widetilde{\nabla}_{\tau e_k}^{\mathcal{F},\beta,\varepsilon}(\tau\sigma) \right\rangle = 0.$$

By (2.90), (2.91) and (2.95)-(2.97), one gets that for any  $x \in s(M)$ ,

$$(2.98) \quad \sum_{i,k=1, i \neq k}^{q_2} \left\langle (1 - p_{T,\beta,\varepsilon}) c_{\beta,\varepsilon}(\tau e_i) \widetilde{\nabla}_{\tau e_i}^{\mathcal{F},\beta,\varepsilon} J_{T,\beta,\varepsilon} \tau \sigma, c_{\beta,\varepsilon}(\tau e_k) \widetilde{\nabla}_{\tau e_k}^{\mathcal{F},\beta,\varepsilon} J_{T,\beta,\varepsilon} \tau \sigma \right\rangle$$

$$= O\left(\varepsilon^2 + \frac{1}{\sqrt{T}}\right) \int_{s(M)} |\sigma|^2 dv_{s(M)} + O\left(\frac{1}{\sqrt{T}}\right) \int_{s(M)} \sum_{i=1}^{q+q_1} \left| Q \nabla_{f_i}^{\mathcal{F},\phi_1(\mathcal{F}_1^\perp),\beta,\varepsilon}(\tau \sigma) \right|^2 dv_{s(M)}.$$

**2.7. Estimates of inner product terms in (2.22), Part II.** In this subsection, we deal with the term left in (2.55). First of all, it is easy to see that the last term in (2.55) verifies the following estimate,

$$(2.99) \quad \left\langle c_{\beta,\varepsilon}(\widetilde{\tau} f_i) c_{\beta,\varepsilon}(\widetilde{\tau} f_j) f_T p_{T,\beta,\varepsilon}(\widetilde{\tau} f_i(f_T) \tau \sigma), {}^Q \widetilde{\nabla}_{\widetilde{\tau} f_j}^{\mathcal{F},\beta,\varepsilon}(\tau \sigma) - \tau \left( {}^Q \widetilde{\nabla}_{\widetilde{\tau} f_j}^{\mathcal{F},\beta,\varepsilon}(\tau \sigma) \Big|_{s(M)} \right) \right\rangle$$

$$= O\left(\frac{1}{\sqrt{T}}\right) \int_{s(M)} |\sigma|^2 dv_{s(M)} + O\left(\frac{1}{\sqrt{T}}\right) \int_{s(M)} \sum_{i=1}^{q+q_1} \left| Q \nabla_{f_i}^{\mathcal{F},\phi_1(\mathcal{F}_1^\perp),\beta,\varepsilon}(\tau \sigma) \right|^2 dv_{s(M)}.$$

Thus we need to deal with the term

$$(2.100) \quad \left\langle c_{\beta,\varepsilon}(\widetilde{\tau} f_i) c_{\beta,\varepsilon}(\widetilde{\tau} f_j) \widetilde{\tau} f_i(f_T) f_T \tau \sigma, {}^Q \widetilde{\nabla}_{\widetilde{\tau} f_j}^{\mathcal{F},\beta,\varepsilon}(\tau \sigma) - \tau \left( {}^Q \widetilde{\nabla}_{\widetilde{\tau} f_j}^{\mathcal{F},\beta,\varepsilon}(\tau \sigma) \Big|_{s(M)} \right) \right\rangle.$$

For any  $(x, Z) \in U_\alpha(\mathcal{F}_2^\perp)$ , one has by Lemma 2.7,

$$(2.101) \quad \left\langle c_{\beta,\varepsilon}(\widetilde{\tau} f_i) c_{\beta,\varepsilon}(\widetilde{\tau} f_j) \tau \sigma, {}^Q \widetilde{\nabla}_{\widetilde{\tau} f_j}^{\mathcal{F},\beta,\varepsilon}(\tau \sigma) - \tau \left( {}^Q \widetilde{\nabla}_{\widetilde{\tau} f_j}^{\mathcal{F},\beta,\varepsilon}(\tau \sigma) \Big|_{s(M)} \right) \right\rangle$$

$$= Z \left\langle c_{\beta,\varepsilon}(\widetilde{\tau} f_i) c_{\beta,\varepsilon}(\widetilde{\tau} f_j) \tau \sigma, {}^Q \widetilde{\nabla}_{\widetilde{\tau} f_j}^{\mathcal{F},\beta,\varepsilon}(\tau \sigma) \right\rangle$$

$$+ O(|Z|^2) \left( |\sigma_x|^2 + \sum_{i=1}^{q+q_1} \left| Q \nabla_{f_i}^{\mathcal{F},\phi_1(\mathcal{F}_1^\perp),\beta,\varepsilon}(\tau \sigma) \right|_x^2 \right).$$

By (2.29),

$$(2.102) \quad Z \left\langle c_{\beta,\varepsilon}(\widetilde{\tau} f_i) c_{\beta,\varepsilon}(\widetilde{\tau} f_j) \tau \sigma, {}^Q \widetilde{\nabla}_{\widetilde{\tau} f_j}^{\mathcal{F},\beta,\varepsilon}(\tau \sigma) \right\rangle$$

$$= \left\langle c_{\beta,\varepsilon}(\widetilde{\tau} f_i) c_{\beta,\varepsilon}(\widetilde{\tau} f_j) \tau \sigma, {}^Q \nabla_Z^{\mathcal{F},\phi_1(\mathcal{F}_1^\perp),\beta,\varepsilon} {}^Q \nabla_{\widetilde{\tau} f_j}^{\mathcal{F},\phi_1(\mathcal{F}_1^\perp),\beta,\varepsilon}(\tau \sigma) \right\rangle$$

$$= \left\langle c_{\beta,\varepsilon}(\widetilde{\tau} f_i) c_{\beta,\varepsilon}(\widetilde{\tau} f_j) \tau \sigma, \left( {}^Q R^{\mathcal{F},\beta,\varepsilon}(Z, \widetilde{\tau} f_j) + {}^Q \nabla_{[Z, \widetilde{\tau} f_j]}^{\mathcal{F},\phi_1(\mathcal{F}_1^\perp),\beta,\varepsilon} \right) \tau \sigma \right\rangle,$$

where  ${}^Q R^{\mathcal{F},\beta,\varepsilon}$  is the curvature of  ${}^Q \widetilde{\nabla}^{\mathcal{F},\beta,\varepsilon}$ .

Clearly,

$$(2.103) \quad {}^Q R^{\mathcal{F},\beta,\varepsilon} = {}^Q R^{\mathcal{F},\phi_1(\mathcal{F}_1^\perp),\beta,\varepsilon} Q - {}^Q \nabla^{\mathcal{F},\phi_1(\mathcal{F}_1^\perp),\beta,\varepsilon} (1 - Q) \nabla^{\mathcal{F},\phi_1(\mathcal{F}_1^\perp),\beta,\varepsilon} Q.$$

Recall that  $f'_1, \dots, f'_{q+q_1}$  is an orthonormal basis of  $\mathcal{F} \oplus \mathcal{F}_1^\perp$  with respect to  $g^\mathcal{F} \oplus g^{\mathcal{F}_1^\perp}$  not depending on  $\beta$  and  $\varepsilon$ , such that  $f'_1, \dots, f'_q$  is an orthonormal basis of  $\mathcal{F}$  verifying (2.33).

By definition (cf.(1.66)), one has

(2.104)

$$\begin{aligned}
(QR^{\mathcal{F}, \phi_1(\mathcal{F}_1^\perp), \beta, \varepsilon} Q)(Z, \tau f_j) &= \frac{1}{4\beta^2} \sum_{s, t=1}^q \langle R^{T\mathcal{M}, \beta, \varepsilon}(Z, \tau f_j) \tau f_s, \tau f_t \rangle_{\beta, \varepsilon} c_{\beta, \varepsilon}(\beta^{-1} \tau f_s) c_{\beta, \varepsilon}(\beta^{-1} \tau f_t) \\
&\quad + \frac{\varepsilon^2}{4} \sum_{s, t=q+1}^{q+q_1} \langle R^{T\mathcal{M}, \beta, \varepsilon}(Z, \tau f_j) \tau f_s, \tau f_t \rangle_{\beta, \varepsilon} c_{\beta, \varepsilon}(\varepsilon \tau f_s) c_{\beta, \varepsilon}(\varepsilon \tau f_t) \\
&\quad + \frac{\varepsilon}{2\beta} \sum_{s=1}^q \sum_{t=q+1}^{q+q_1} \langle R^{T\mathcal{M}, \beta, \varepsilon}(Z, \tau f_j) \tau f_s, \tau f_t \rangle_{\beta, \varepsilon} c_{\beta, \varepsilon}(\beta^{-1} \tau f_s) c_{\beta, \varepsilon}(\varepsilon \tau f_t) \\
&\quad - \frac{\varepsilon^2}{4} \sum_{s, t=q+1}^{q+q_1} \left\langle R^{\mathcal{F}_1^\perp, \beta, \varepsilon}(Z, \tau f_j) f'_s, f'_t \right\rangle_{\beta, \varepsilon} \widehat{c}_{\beta, \varepsilon}(\varepsilon f'_s) \widehat{c}_{\beta, \varepsilon}(\varepsilon f'_t) + R^{\phi_1(\mathcal{F}_1^\perp), \beta, \varepsilon}(Z, \tau f_j).
\end{aligned}$$

If  $1 \leq j, s, t \leq q$ , one verifies, by (2.41) that

$$\begin{aligned}
(2.105) \quad \frac{1}{\beta^2} \langle R^{T\mathcal{M}, \beta, \varepsilon}(Z, \tau f_j) \tau f_s, \tau f_t \rangle_{\beta, \varepsilon} &= \langle R^{T\mathcal{M}, \beta, \varepsilon}(f'_s, f'_t) Z, f'_j \rangle + O(|Z|^2) \\
&= \langle \nabla_{f'_s}^{T\mathcal{M}, \beta, \varepsilon} \nabla_{f'_t}^{T\mathcal{M}, \beta, \varepsilon} Z, f'_j \rangle - \langle \nabla_{f'_t}^{T\mathcal{M}, \beta, \varepsilon} \nabla_{f'_s}^{T\mathcal{M}, \beta, \varepsilon} Z, f'_j \rangle - \langle \nabla_{[f'_s, f'_t]}^{T\mathcal{M}, \beta, \varepsilon} Z, f'_j \rangle + O(|Z|^2) \\
&= -\langle p \nabla_{f'_t}^{T\mathcal{M}, \beta, \varepsilon} Z, \nabla_{f'_s}^{T\mathcal{M}, \beta, \varepsilon} f'_j \rangle - \frac{1}{\beta^2 \varepsilon^2} \langle p_1^\perp \nabla_{f'_t}^{T\mathcal{M}, \beta, \varepsilon} Z, \nabla_{f'_s}^{T\mathcal{M}, \beta, \varepsilon} f'_j \rangle - \frac{1}{\beta^2} \langle p_2^\perp \nabla_{f'_t}^{T\mathcal{M}, \beta, \varepsilon} Z, \nabla_{f'_s}^{T\mathcal{M}, \beta, \varepsilon} f'_j \rangle \\
&\quad + \langle p \nabla_{f'_s}^{T\mathcal{M}, \beta, \varepsilon} Z, \nabla_{f'_t}^{T\mathcal{M}, \beta, \varepsilon} f'_j \rangle + \frac{1}{\beta^2 \varepsilon^2} \langle p_1^\perp \nabla_{f'_s}^{T\mathcal{M}, \beta, \varepsilon} Z, \nabla_{f'_t}^{T\mathcal{M}, \beta, \varepsilon} f'_j \rangle + \frac{1}{\beta^2} \langle p_2^\perp \nabla_{f'_s}^{T\mathcal{M}, \beta, \varepsilon} Z, \nabla_{f'_t}^{T\mathcal{M}, \beta, \varepsilon} f'_j \rangle \\
&\quad + f'_s \left( \langle \nabla_{f'_t}^{T\mathcal{M}, \beta, \varepsilon} Z, f'_j \rangle \right) - f'_t \left( \langle \nabla_{f'_s}^{T\mathcal{M}, \beta, \varepsilon} Z, f'_j \rangle \right) - \langle \nabla_{[f'_s, f'_t]}^{T\mathcal{M}, \beta, \varepsilon} Z, f'_j \rangle + O(|Z|^2) \\
&= O(\varepsilon^2 |Z|) + O(|Z|^2).
\end{aligned}$$

If  $1 \leq j \leq q$  and  $q+1 \leq s, t \leq q+q_1$ , one has, in view of (1.19),

$$\begin{aligned}
(2.106) \quad \varepsilon^2 \langle R^{T\mathcal{M}, \beta, \varepsilon}(Z, \tau f_j) \tau f_s, \tau f_t \rangle_{\beta, \varepsilon} &= \beta^2 \varepsilon^2 \langle R^{T\mathcal{M}, \beta, \varepsilon}(f'_s, f'_t) Z, f'_j \rangle + O(|Z|^2) \\
&= \beta^2 \varepsilon^2 \langle \nabla_{f'_s}^{T\mathcal{M}, \beta, \varepsilon} \nabla_{f'_t}^{T\mathcal{M}, \beta, \varepsilon} Z, f'_j \rangle - \beta^2 \varepsilon^2 \langle \nabla_{f'_t}^{T\mathcal{M}, \beta, \varepsilon} \nabla_{f'_s}^{T\mathcal{M}, \beta, \varepsilon} Z, f'_j \rangle - \beta^2 \varepsilon^2 \langle \nabla_{[f'_s, f'_t]}^{T\mathcal{M}, \beta, \varepsilon} Z, f'_j \rangle \\
&= -\beta^2 \varepsilon^2 \langle p \nabla_{f'_t}^{T\mathcal{M}, \beta, \varepsilon} Z, \nabla_{f'_s}^{T\mathcal{M}, \beta, \varepsilon} f'_j \rangle - \langle p_1^\perp \nabla_{f'_t}^{T\mathcal{M}, \beta, \varepsilon} Z, \nabla_{f'_s}^{T\mathcal{M}, \beta, \varepsilon} f'_j \rangle - \varepsilon^2 \langle p_2^\perp \nabla_{f'_t}^{T\mathcal{M}, \beta, \varepsilon} Z, \nabla_{f'_s}^{T\mathcal{M}, \beta, \varepsilon} f'_j \rangle \\
&\quad + \beta^2 \varepsilon^2 \langle p \nabla_{f'_s}^{T\mathcal{M}, \beta, \varepsilon} Z, \nabla_{f'_t}^{T\mathcal{M}, \beta, \varepsilon} f'_j \rangle + \langle p_1^\perp \nabla_{f'_s}^{T\mathcal{M}, \beta, \varepsilon} Z, \nabla_{f'_t}^{T\mathcal{M}, \beta, \varepsilon} f'_j \rangle + \varepsilon^2 \langle p_2^\perp \nabla_{f'_s}^{T\mathcal{M}, \beta, \varepsilon} Z, \nabla_{f'_t}^{T\mathcal{M}, \beta, \varepsilon} f'_j \rangle \\
&\quad + \beta^2 \varepsilon^2 f'_s \left( \langle \nabla_{f'_t}^{T\mathcal{M}, \beta, \varepsilon} Z, f'_j \rangle \right) - \beta^2 \varepsilon^2 f'_t \left( \langle \nabla_{f'_s}^{T\mathcal{M}, \beta, \varepsilon} Z, f'_j \rangle \right) - \beta^2 \varepsilon^2 \langle \nabla_{[f'_s, f'_t]}^{T\mathcal{M}, \beta, \varepsilon} Z, f'_j \rangle + O(|Z|^2) \\
&= O(\varepsilon^2 |Z|) + O(|Z|^2).
\end{aligned}$$

If  $1 \leq j, t \leq q$  and  $q+1 \leq s \leq q+q_1$ , one has

$$\begin{aligned}
(2.107) \quad \frac{\varepsilon}{\beta} \langle R^{T\mathcal{M}, \beta, \varepsilon}(Z, \tau f_j) \tau f_s, \tau f_t \rangle_{\beta, \varepsilon} &= \beta \varepsilon \langle R^{T\mathcal{M}, \beta, \varepsilon}(f'_s, f'_t) Z, f'_j \rangle + O(|Z|^2) \\
&= \beta \varepsilon \langle \nabla_{f'_s}^{T\mathcal{M}, \beta, \varepsilon} \nabla_{f'_t}^{T\mathcal{M}, \beta, \varepsilon} Z, f'_j \rangle - \beta \varepsilon \langle \nabla_{f'_t}^{T\mathcal{M}, \beta, \varepsilon} \nabla_{f'_s}^{T\mathcal{M}, \beta, \varepsilon} Z, f'_j \rangle - \beta \varepsilon \langle \nabla_{[f'_s, f'_t]}^{T\mathcal{M}, \beta, \varepsilon} Z, f'_j \rangle
\end{aligned}$$

$$\begin{aligned}
&= -\beta\varepsilon \left\langle p\nabla_{f'_t}^{TM,\beta,\varepsilon} Z, \nabla_{f'_s}^{TM,\beta,\varepsilon} f'_j \right\rangle - \frac{1}{\beta\varepsilon} \left\langle p_1^\perp \nabla_{f'_t}^{TM,\beta,\varepsilon} Z, \nabla_{f'_s}^{TM,\beta,\varepsilon} f'_j \right\rangle - \frac{\varepsilon}{\beta} \left\langle p_2^\perp \nabla_{f'_t}^{TM,\beta,\varepsilon} Z, \nabla_{f'_s}^{TM,\beta,\varepsilon} f'_j \right\rangle \\
&+ \beta\varepsilon \left\langle p\nabla_{f'_s}^{TM,\beta,\varepsilon} Z, \nabla_{f'_t}^{TM,\beta,\varepsilon} f'_j \right\rangle + \frac{1}{\beta\varepsilon} \left\langle p_1^\perp \nabla_{f'_s}^{TM,\beta,\varepsilon} Z, \nabla_{f'_t}^{TM,\beta,\varepsilon} f'_j \right\rangle + \frac{\varepsilon}{\beta} \left\langle p_2^\perp \nabla_{f'_s}^{TM,\beta,\varepsilon} Z, \nabla_{f'_t}^{TM,\beta,\varepsilon} f'_j \right\rangle \\
&+ \beta\varepsilon f'_s \left( \left\langle \nabla_{f'_t}^{TM,\beta,\varepsilon} Z, f'_j \right\rangle \right) - \beta\varepsilon f'_t \left( \left\langle \nabla_{f'_s}^{TM,\beta,\varepsilon} Z, f'_j \right\rangle \right) - \beta\varepsilon \left\langle \nabla_{[f'_s, f'_t]}^{TM,\beta,\varepsilon} Z, f'_j \right\rangle + O(|Z|^2) = O\left(\frac{\varepsilon|Z|}{\beta}\right) + O(|Z|^2).
\end{aligned}$$

If  $q+1 \leq j \leq q+q_1$  and  $1 \leq s, t \leq q$ , one has

$$\begin{aligned}
(2.108) \quad &\frac{1}{\beta^2} \left\langle R^{TM,\beta,\varepsilon}(Z, \tau f_j) \tau f_s, \tau f_t \right\rangle_{\beta,\varepsilon} = \frac{1}{\beta^2 \varepsilon^2} \left\langle R^{TM,\beta,\varepsilon}(f'_s, f'_t) Z, f'_j \right\rangle + O(|Z|^2) \\
&= \frac{1}{\beta^2 \varepsilon^2} \left\langle \nabla_{f'_s}^{TM,\beta,\varepsilon} \nabla_{f'_t}^{TM,\beta,\varepsilon} Z, f'_j \right\rangle - \frac{1}{\beta^2 \varepsilon^2} \left\langle \nabla_{f'_t}^{TM,\beta,\varepsilon} \nabla_{f'_s}^{TM,\beta,\varepsilon} Z, f'_j \right\rangle - \frac{1}{\beta^2 \varepsilon^2} \left\langle \nabla_{[f'_s, f'_t]}^{TM,\beta,\varepsilon} Z, f'_j \right\rangle \\
&= - \left\langle p\nabla_{f'_t}^{TM,\beta,\varepsilon} Z, \nabla_{f'_s}^{TM,\beta,\varepsilon} f'_j \right\rangle - \frac{1}{\beta^2 \varepsilon^2} \left\langle p_1^\perp \nabla_{f'_t}^{TM,\beta,\varepsilon} Z, \nabla_{f'_s}^{TM,\beta,\varepsilon} f'_j \right\rangle - \frac{1}{\beta^2} \left\langle p_2^\perp \nabla_{f'_t}^{TM,\beta,\varepsilon} Z, \nabla_{f'_s}^{TM,\beta,\varepsilon} f'_j \right\rangle \\
&+ \left\langle p\nabla_{f'_s}^{TM,\beta,\varepsilon} Z, \nabla_{f'_t}^{TM,\beta,\varepsilon} f'_j \right\rangle + \frac{1}{\beta^2 \varepsilon^2} \left\langle p_1^\perp \nabla_{f'_s}^{TM,\beta,\varepsilon} Z, \nabla_{f'_t}^{TM,\beta,\varepsilon} f'_j \right\rangle + \frac{1}{\beta^2} \left\langle p_2^\perp \nabla_{f'_s}^{TM,\beta,\varepsilon} Z, \nabla_{f'_t}^{TM,\beta,\varepsilon} f'_j \right\rangle \\
&+ \frac{1}{\beta^2 \varepsilon^2} f'_s \left( \left\langle \nabla_{f'_t}^{TM,\beta,\varepsilon} Z, f'_j \right\rangle \right) - \frac{1}{\beta^2 \varepsilon^2} f'_t \left( \left\langle \nabla_{f'_s}^{TM,\beta,\varepsilon} Z, f'_j \right\rangle \right) - \frac{1}{\beta^2 \varepsilon^2} \left\langle \nabla_{[f'_s, f'_t]}^{TM,\beta,\varepsilon} Z, f'_j \right\rangle + O(|Z|^2) \\
&= O\left(\frac{|Z|}{\beta^2}\right) + O(|Z|^2).
\end{aligned}$$

If  $q+1 \leq j, s, t \leq q+q_1$ , one has

$$\begin{aligned}
(2.109) \quad &\varepsilon^2 \left\langle R^{TM,\beta,\varepsilon}(Z, \tau f_j) \tau f_s, \tau f_t \right\rangle_{\beta,\varepsilon} = \left\langle R^{TM,\beta,\varepsilon}(f'_s, f'_t) Z, f'_j \right\rangle + O(|Z|^2) \\
&= \left\langle \nabla_{f'_s}^{TM,\beta,\varepsilon} \nabla_{f'_t}^{TM,\beta,\varepsilon} Z, f'_j \right\rangle - \left\langle \nabla_{f'_t}^{TM,\beta,\varepsilon} \nabla_{f'_s}^{TM,\beta,\varepsilon} Z, f'_j \right\rangle - \left\langle \nabla_{[f'_s, f'_t]}^{TM,\beta,\varepsilon} Z, f'_j \right\rangle \\
&= -\beta^2 \varepsilon^2 \left\langle p\nabla_{f'_t}^{TM,\beta,\varepsilon} Z, \nabla_{f'_s}^{TM,\beta,\varepsilon} f'_j \right\rangle - \left\langle p_1^\perp \nabla_{f'_t}^{TM,\beta,\varepsilon} Z, \nabla_{f'_s}^{TM,\beta,\varepsilon} f'_j \right\rangle - \varepsilon^2 \left\langle p_2^\perp \nabla_{f'_t}^{TM,\beta,\varepsilon} Z, \nabla_{f'_s}^{TM,\beta,\varepsilon} f'_j \right\rangle \\
&+ \beta^2 \varepsilon^2 \left\langle p\nabla_{f'_s}^{TM,\beta,\varepsilon} Z, \nabla_{f'_t}^{TM,\beta,\varepsilon} f'_j \right\rangle + \left\langle p_1^\perp \nabla_{f'_s}^{TM,\beta,\varepsilon} Z, \nabla_{f'_t}^{TM,\beta,\varepsilon} f'_j \right\rangle + \varepsilon^2 \left\langle p_2^\perp \nabla_{f'_s}^{TM,\beta,\varepsilon} Z, \nabla_{f'_t}^{TM,\beta,\varepsilon} f'_j \right\rangle \\
&+ f'_s \left( \left\langle \nabla_{f'_t}^{TM,\beta,\varepsilon} Z, f'_j \right\rangle \right) - f'_t \left( \left\langle \nabla_{f'_s}^{TM,\beta,\varepsilon} Z, f'_j \right\rangle \right) - \left\langle \nabla_{[f'_s, f'_t]}^{TM,\beta,\varepsilon} Z, f'_j \right\rangle + O(|Z|^2) \\
&= O(|Z|) + O(|Z|^2).
\end{aligned}$$

If  $q+1 \leq j, t \leq q+q_1$  and  $1 \leq s \leq q$ , one has

$$\begin{aligned}
(2.110) \quad &-\frac{\varepsilon}{\beta} \left\langle R^{TM,\beta,\varepsilon}(Z, \tau f_j) \tau f_s, \tau f_t \right\rangle_{\beta,\varepsilon} = \beta\varepsilon \left\langle R^{TM,\beta,\varepsilon}(Z, f'_j) f'_t, f'_s \right\rangle + O(|Z|^2) \\
&= \beta\varepsilon \left\langle \nabla_Z^{TM,\beta,\varepsilon} \nabla_{f'_j}^{TM,\beta,\varepsilon} f'_t, f'_s \right\rangle - \beta\varepsilon \left\langle \nabla_{f'_j}^{TM,\beta,\varepsilon} \nabla_Z^{TM,\beta,\varepsilon} f'_t, f'_s \right\rangle - \beta\varepsilon \left\langle \nabla_{[Z, f'_j]}^{TM,\beta,\varepsilon} f'_t, f'_s \right\rangle + O(|Z|^2) \\
&= -\beta\varepsilon \left\langle p\nabla_{f'_j}^{TM,\beta,\varepsilon} f'_t, \nabla_Z^{TM,\beta,\varepsilon} f'_s \right\rangle - \frac{1}{\beta\varepsilon} \left\langle p_1^\perp \nabla_{f'_j}^{TM,\beta,\varepsilon} f'_t, \nabla_Z^{TM,\beta,\varepsilon} f'_s \right\rangle - \frac{\varepsilon}{\beta} \left\langle p_2^\perp \nabla_{f'_j}^{TM,\beta,\varepsilon} f'_t, \nabla_Z^{TM,\beta,\varepsilon} f'_s \right\rangle \\
&+ \beta\varepsilon \left\langle p\nabla_Z^{TM,\beta,\varepsilon} f'_t, \nabla_{f'_j}^{TM,\beta,\varepsilon} f'_s \right\rangle + \frac{1}{\beta\varepsilon} \left\langle p_1^\perp \nabla_Z^{TM,\beta,\varepsilon} f'_t, \nabla_{f'_j}^{TM,\beta,\varepsilon} f'_s \right\rangle + \frac{\varepsilon}{\beta} \left\langle p_2^\perp \nabla_Z^{TM,\beta,\varepsilon} f'_t, \nabla_{f'_j}^{TM,\beta,\varepsilon} f'_s \right\rangle \\
&+ \beta\varepsilon Z \left( \left\langle \nabla_{f'_j}^{TM,\beta,\varepsilon} f'_t, f'_s \right\rangle \right) - \beta\varepsilon f'_j \left( \left\langle \nabla_Z^{TM,\beta,\varepsilon} f'_t, f'_s \right\rangle \right) - \beta\varepsilon \left\langle \nabla_{[Z, f'_j]}^{TM,\beta,\varepsilon} f'_t, f'_s \right\rangle + O(|Z|^2) \\
&= O\left(\frac{\varepsilon|Z|}{\beta}\right) + O(|Z|^2).
\end{aligned}$$

If  $1 \leq j \leq q + q_1$  and  $q + 1 \leq s, t \leq q + q_1$ , one has

$$\begin{aligned}
 (2.111) \quad & \left\langle R^{\mathcal{F}_1^\perp, \beta, \varepsilon}(Z, \tau f_j) f'_t, f'_s \right\rangle \\
 &= \left\langle \nabla_Z^{\mathcal{F}_1^\perp, \beta, \varepsilon} \nabla_{f'_j}^{\mathcal{F}_1^\perp, \beta, \varepsilon} f'_t, f'_s \right\rangle - \left\langle \nabla_{f'_j}^{\mathcal{F}_1^\perp, \beta, \varepsilon} \nabla_Z^{\mathcal{F}_1^\perp, \beta, \varepsilon} f'_t, f'_s \right\rangle - \left\langle \nabla_{[Z, f'_j]}^{\mathcal{F}_1^\perp, \beta, \varepsilon} f'_t, f'_s \right\rangle + O(|Z|^2) \\
 &= \left\langle R^{T\mathcal{M}, \beta, \varepsilon}(Z, f'_j) f'_t, f'_s \right\rangle + \beta^2 \varepsilon^2 \left\langle p \nabla_{f'_j}^{T\mathcal{M}, \beta, \varepsilon} f'_t, \nabla_Z^{T\mathcal{M}, \beta, \varepsilon} f'_s \right\rangle + \varepsilon^2 \left\langle p_2^\perp \nabla_{f'_j}^{T\mathcal{M}, \beta, \varepsilon} f'_t, \nabla_Z^{T\mathcal{M}, \beta, \varepsilon} f'_s \right\rangle \\
 &\quad - \beta^2 \varepsilon^2 \left\langle p \nabla_Z^{T\mathcal{M}, \beta, \varepsilon} f'_t, \nabla_{f'_j}^{T\mathcal{M}, \beta, \varepsilon} f'_s \right\rangle - \varepsilon^2 \left\langle p_2^\perp \nabla_Z^{T\mathcal{M}, \beta, \varepsilon} f'_t, \nabla_{f'_j}^{T\mathcal{M}, \beta, \varepsilon} f'_s \right\rangle + O(|Z|^2).
 \end{aligned}$$

By (2.106) and (2.111), one sees that when  $1 \leq j \leq q, q + 1 \leq s, t \leq q + q_1$ , one has

$$(2.112) \quad \left\langle R^{\mathcal{F}_1^\perp, \beta, \varepsilon}(Z, \tau f_j) f'_t, f'_s \right\rangle = O(\varepsilon^2 |Z|) + O(|Z|^2),$$

while by (2.109) and (2.111), one sees that when  $q + 1 \leq j \leq q + q_1, q + 1 \leq s, t \leq q + q_1$ , one has

$$(2.113) \quad \left\langle R^{\mathcal{F}_1^\perp, \beta, \varepsilon}(Z, \tau f_j) f'_t, f'_s \right\rangle = O(|Z|) + O(|Z|^2).$$

Now from (2.85)-(2.87), one verifies easily that

$$(2.114) \quad (1 - Q) \nabla_Z^{\mathcal{F}, \phi_1(\mathcal{F}_1^\perp), \beta, \varepsilon} Q = O(\varepsilon |Z|) + O(|Z|^2).$$

Similarly, one has

$$(2.115) \quad Q \nabla_Z^{\mathcal{F}, \phi_1(\mathcal{F}_1^\perp), \beta, \varepsilon} (1 - Q) = O(\varepsilon |Z|) + O(|Z|^2).$$

On the other hand, by (2.57)-(2.59), one finds that for  $1 \leq j \leq q$ ,

$$(2.116) \quad (1 - Q) \nabla_{\tau f_j}^{\mathcal{F}, \phi_1(\mathcal{F}_1^\perp), \beta, \varepsilon} Q = O(\varepsilon) + O_{\beta, \varepsilon}(|Z|).$$

Similarly,

$$(2.117) \quad Q \nabla_{\tau f_j}^{\mathcal{F}, \phi_1(\mathcal{F}_1^\perp), \beta, \varepsilon} (1 - Q) = O(\varepsilon) + O_{\beta, \varepsilon}(|Z|).$$

While for  $q + 1 \leq j \leq q + q_1$ , by (2.57), (2.61) and (2.62), one has

$$(2.118) \quad (1 - Q) \nabla_{\tau f_j}^{\mathcal{F}, \phi_1(\mathcal{F}_1^\perp), \beta, \varepsilon} Q = O(\beta^{-1} + \varepsilon^{-1}) + O_{\beta, \varepsilon}(|Z|).$$

Similarly,

$$(2.119) \quad Q \nabla_{\tau f_j}^{\mathcal{F}, \phi_1(\mathcal{F}_1^\perp), \beta, \varepsilon} (1 - Q) = O(\beta^{-1} + \varepsilon^{-1}) + O_{\beta, \varepsilon}(|Z|).$$

From (2.103)-(2.119), one gets that if  $1 \leq i, j \leq q + q_1$  then the following identity holds at  $(x, Z)$  near  $s(M)$ ,

$$(2.120) \quad \left\langle c_{\beta, \varepsilon}(\tilde{\tau} f_i) c_{\beta, \varepsilon}(\tilde{\tau} f_j) {}^Q R^{\mathcal{F}, \phi_1(\mathcal{F}_1^\perp), \beta, \varepsilon}(Z, \tilde{\tau} f_j) \tau \sigma, \tau \sigma \right\rangle = O\left(\frac{\varepsilon}{\beta^2} |Z| + |Z|^2\right) |\sigma|^2.$$

Now we examine the term

$$\left\langle c_{\beta, \varepsilon}(\tilde{\tau} f_i) c_{\beta, \varepsilon}(\tilde{\tau} f_j) \tau \sigma, {}^Q \nabla_{[Z, \tilde{\tau} f_j]}^{\mathcal{F}, \phi_1(\mathcal{F}_1^\perp), \beta, \varepsilon}(\tau \sigma) \right\rangle$$

in (2.102).

Write  $Z = \sum_{k=1}^{q_2} z_k \tau e_k$ . Then one has, by (2.37),

$$(2.121) \quad (p + p_1^\perp) [Z, \tau f_j] = - (p + p_1^\perp) \nabla_{\tau f_j}^{T\mathcal{M}, \beta, \varepsilon} Z = - \sum_{k=1}^{q_2} z_k (p + p_1^\perp) \nabla_{\tau f_j}^{T\mathcal{M}, \beta, \varepsilon} (\tau e_k).$$

For any  $1 \leq k \leq q_2$ ,  $1 \leq j \leq q$ , by (2.41) one verifies easily that

$$(2.122) \quad \begin{aligned} (p + p_1^\perp) \nabla_{\tau f_j}^{T\mathcal{M}, \beta, \varepsilon} (\tau e_k) &= \sum_{s=1}^q \left\langle \nabla_{\tau f_j}^{T\mathcal{M}, \beta, \varepsilon} (\tau e_k), f'_s \right\rangle f'_s + \sum_{s=q+1}^{q+q_1} \left\langle \nabla_{\tau f_j}^{T\mathcal{M}, \beta, \varepsilon} (\tau e_k), f'_s \right\rangle f'_s \\ &= \sum_{s=1}^q O_{\beta, \varepsilon}(|Z|) f'_s + \sum_{s=q+1}^{q+q_1} (O(\varepsilon^2) + O_{\beta, \varepsilon}(|Z|)) f'_s. \end{aligned}$$

By (2.121) and (2.122), for  $1 \leq j \leq q$ , one has,

$$(2.123) \quad \begin{aligned} \frac{1}{\beta} Q \nabla_{(p+p_1^\perp)[Z, \tau f_j]}^{\mathcal{F}, \phi_1(\mathcal{F}_1^\perp), \beta, \varepsilon} (\tau \sigma) &= \sum_{i=1}^q O(|Z|^2) Q \nabla_{f'_i}^{\mathcal{F}, \phi_1(\mathcal{F}_1^\perp), \beta, \varepsilon} (\tau \sigma) \\ &\quad + \sum_{i=q+1}^{q+q_1} O\left(\frac{\varepsilon^2 |Z|}{\beta} + |Z|^2\right) Q \nabla_{f'_i}^{\mathcal{F}, \phi_1(\mathcal{F}_1^\perp), \beta, \varepsilon} (\tau \sigma). \end{aligned}$$

Similarly, for  $1 \leq k \leq q_2$ ,  $q+1 \leq j \leq q+q_1$ , one has

$$(2.124) \quad p \nabla_{\tau f_j}^{T\mathcal{M}, \beta, \varepsilon} (\tau e_k) = \sum_{s=1}^q \left\langle \nabla_{\tau f_j}^{T\mathcal{M}, \beta, \varepsilon} (\tau e_k), f'_s \right\rangle f'_s = \sum_{s=1}^q O(\beta^{-2}) f'_s + \sum_{s=1}^q O_{\beta, \varepsilon}(|Z|) f'_s.$$

Thus, for  $q+1 \leq j \leq q+q_1$ , one has,

$$(2.125) \quad \varepsilon Q \nabla_{p[Z, \tau f_j]}^{\mathcal{F}, \phi_1(\mathcal{F}_1^\perp), \beta, \varepsilon} (\tau \sigma) = \sum_{i=1}^q O\left(\frac{\varepsilon |Z|}{\beta^2} + |Z|^2\right) Q \nabla_{f'_i}^{\mathcal{F}, \phi_1(\mathcal{F}_1^\perp), \beta, \varepsilon} (\tau \sigma).$$

For  $1 \leq k \leq q_2$ ,  $q+1 \leq j \leq q+q_1$ , one has<sup>10</sup>

$$(2.126) \quad \begin{aligned} p_1^\perp \nabla_{\tau f_j}^{T\mathcal{M}, \beta, \varepsilon} (\tau e_k) &= \sum_{s=q+1}^{q+q_1} \left\langle \nabla_{f'_j}^{T\mathcal{M}, \beta, \varepsilon} (\tau e_k), f'_s \right\rangle f'_s + O_{\beta, \varepsilon}(|Z|) \\ &= -\varepsilon^2 \sum_{s=q+q_1}^{q+1} \left\langle \tau e_k, \nabla_{f'_j}^{T\mathcal{M}, \beta, \varepsilon} f'_s \right\rangle f'_s + O_{\beta, \varepsilon}(|Z|). \end{aligned}$$

Now for any  $1 \leq j \leq q+q_1$ , one has

$$(2.127) \quad \begin{aligned} p_2^\perp [Z, \tau f_j] &= p_2^\perp \nabla_Z^{T\mathcal{M}} (\tau f_j) - \nabla_{\tau f_j}^{\mathcal{F}_2^\perp} Z \\ &= \sum_{k=1}^{q_2} \left\langle \nabla_Z^{T\mathcal{M}} (\tau f_j), \tau e_k \right\rangle \tau e_k - \sum_{k=1}^{q_2} \tau f_j(z_k) \tau e_k - \sum_{k=1}^{q_2} z_k \nabla_{\tau f_j}^{\mathcal{F}_2^\perp} (\tau e_k). \end{aligned}$$

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<sup>10</sup>This formula will not be used later as we can't find a suitable estimate of the right hand side.

From (2.127) and Lemmas 2.7, 2.8, one finds

$$(2.128) \quad Q \nabla_{p_2^\perp[Z, \tau f_j]}^{\mathcal{F}, \phi_1(\mathcal{F}_1^\perp), \beta, \varepsilon}(\tau \sigma) = - \sum_{k=1}^{q_2} \tau f_j(z_k) Q \nabla_{\tau e_k}^{\mathcal{F}, \phi_1(\mathcal{F}_1^\perp), \beta, \varepsilon}(\tau \sigma) \\ + O(|Z|^2) \left( |\sigma|_x + \sum_{k=1}^{q+q_1} \left| Q \nabla_{f_k}^{\mathcal{F}, \phi_1(\mathcal{F}_1^\perp), \beta, \varepsilon}(\tau \sigma) \right|_x \right).$$

For another section  $s$  on  $s(M)$ , one has

$$(2.129) \quad Z \left\langle Q \nabla_{\tau e_k}^{\mathcal{F}, \phi_1(\mathcal{F}_1^\perp), \beta, \varepsilon}(\tau \sigma), \tau s \right\rangle = \left\langle Q \nabla_Z^{\mathcal{F}, \phi_1(\mathcal{F}_1^\perp), \beta, \varepsilon} Q \nabla_{\tau e_k}^{\mathcal{F}, \phi_1(\mathcal{F}_1^\perp), \beta, \varepsilon}(\tau \sigma), \tau s \right\rangle \\ = \left\langle Q R^{\mathcal{F}, \phi_1(\mathcal{F}_1^\perp), \beta, \varepsilon}(Z, \tau e_k) \tau \sigma, \tau s \right\rangle + \left\langle Q \nabla_{[Z, \tau e_k]}^{\mathcal{F}, \phi_1(\mathcal{F}_1^\perp), \beta, \varepsilon}(\tau \sigma), \tau s \right\rangle.$$

As in (2.127), one verifies

$$(2.130) \quad [Z, \tau e_k] = -\nabla_{\tau e_k}^{\mathcal{F}_2^\perp} Z = - \sum_{j=1}^{q_2} \tau e_k(z_j) \tau e_j - \sum_{j=1}^{q_2} z_j \nabla_{\tau e_k}^{\mathcal{F}_2^\perp}(\tau e_j).$$

Clearly,

$$(2.131) \quad \tau e_k(z_j) = \delta_{kj} + O(|Z|).$$

By Lemma 2.8 and (2.129)-(2.131), one deduces that

$$(2.132) \quad \left\langle Q \nabla_{\tau e_k}^{\mathcal{F}, \phi_1(\mathcal{F}_1^\perp), \beta, \varepsilon}(\tau \sigma), \tau s \right\rangle = \frac{1}{2} \left\langle Q R^{\mathcal{F}, \phi_1(\mathcal{F}_1^\perp), \beta, \varepsilon}(Z, \tau e_k) \tau \sigma, \tau s \right\rangle + O(|Z|^2) \\ = \frac{1}{2} \sum_{m=1}^{q_2} z_m \left\langle Q R^{\mathcal{F}, \phi_1(\mathcal{F}_1^\perp), \beta, \varepsilon}(\tau e_m, \tau e_k) \tau \sigma, \tau s \right\rangle + O(|Z|^2).$$

From (2.128) and (2.132), one gets

$$(2.133) \quad \left\langle c_{\beta, \varepsilon}(\tilde{\tau} f_i) c_{\beta, \varepsilon}(\tilde{\tau} f_j) \tau \sigma, Q \nabla_{p_2^\perp[Z, \tilde{\tau} f_j]}^{\mathcal{F}, \phi_1(\mathcal{F}_1^\perp), \beta, \varepsilon}(\tau \sigma) \right\rangle_{(x, Z)} \\ = -\frac{1}{2} \left\langle c_{\beta, \varepsilon}(\tilde{\tau} f_i) c_{\beta, \varepsilon}(\tilde{\tau} f_j) \tau \sigma, Q R^{\mathcal{F}, \phi_1(\mathcal{F}_1^\perp), \beta, \varepsilon} \left( Z, \nabla_{\tilde{\tau} f_j}^{\mathcal{F}_2^\perp} Z \right) \tau \sigma \right\rangle_{(x, Z)} + O(|Z|^2).$$

From (2.103), (2.114), (2.115) and (2.133), one gets

$$(2.134) \quad \left\langle c_{\beta, \varepsilon}(\tilde{\tau} f_i) c_{\beta, \varepsilon}(\tilde{\tau} f_j) \tau \sigma, Q \nabla_{p_2^\perp[Z, \tilde{\tau} f_j]}^{\mathcal{F}, \phi_1(\mathcal{F}_1^\perp), \beta, \varepsilon}(\tau \sigma) \right\rangle_{(x, Z)} \\ = -\frac{1}{2} \left\langle c_{\beta, \varepsilon}(\tilde{\tau} f_i) c_{\beta, \varepsilon}(\tilde{\tau} f_j) \tau \sigma, R^{\mathcal{F}, \phi_1(\mathcal{F}_1^\perp), \beta, \varepsilon} \left( Z, \nabla_{\tilde{\tau} f_j}^{\mathcal{F}_2^\perp} Z \right) \tau \sigma \right\rangle_{(x, Z)} + O\left(\frac{\varepsilon^2 |Z|}{|\tau f_j|_{\beta, \varepsilon}}\right) + O(|Z|^2).$$

As in (2.104), we have

$$(2.135) \quad \left( Q R^{\mathcal{F}, \phi_1(\mathcal{F}_1^\perp), \beta, \varepsilon} Q \right) (\tau e_m, \tau e_k) \\ = \frac{1}{4\beta^2} \sum_{s, t=1}^q \left\langle R^{T\mathcal{M}, \beta, \varepsilon}(\tau e_m, \tau e_k) \tau f_s, \tau f_t \right\rangle_{\beta, \varepsilon} c_{\beta, \varepsilon}(\beta^{-1} \tau f_s) c_{\beta, \varepsilon}(\beta^{-1} \tau f_t)$$



$$\begin{aligned}
& + \frac{\varepsilon^2}{4} \sum_{s,t=q+1}^{q+q_1} \langle R^{TM,\beta,\varepsilon}(\tau e_m, \tau e_k) \tau f_s, \tau f_t \rangle_{\beta,\varepsilon} c_{\beta,\varepsilon}(\varepsilon \tau f_s) c_{\beta,\varepsilon}(\varepsilon \tau f_t) \\
& + \frac{\varepsilon}{2\beta} \sum_{s=1}^q \sum_{t=q+1}^{q+q_1} \langle R^{TM,\beta,\varepsilon}(\tau e_m, \tau e_k) \tau f_s, \tau f_t \rangle_{\beta,\varepsilon} c_{\beta,\varepsilon}(\beta^{-1} \tau f_s) c_{\beta,\varepsilon}(\varepsilon \tau f_t) \\
& - \frac{\varepsilon^2}{4} \sum_{s,t=q+1}^{q+q_1} \left\langle R^{\mathcal{F}_1^\perp,\beta,\varepsilon}(\tau e_m, \tau e_k) f'_s, f'_t \right\rangle_{\beta,\varepsilon} \widehat{c}_{\beta,\varepsilon}(\varepsilon f'_s) \widehat{c}_{\beta,\varepsilon}(\varepsilon f'_t) + R^{\phi_1(\mathcal{F}_1^\perp),\beta,\varepsilon}(\tau e_m, \tau e_k).
\end{aligned}$$

If  $1 \leq s, t \leq q$ , one has, in view of (2.41) and (2.71), that

$$\begin{aligned}
(2.136) \quad & \frac{1}{\beta^2} \langle R^{TM,\beta,\varepsilon}(\tau e_m, \tau e_k) \tau f_s, \tau f_t \rangle_{\beta,\varepsilon} = \langle R^{TM,\beta,\varepsilon}(\tau e_m, \tau e_k) f'_s, f'_t \rangle + O_{\beta,\varepsilon}(|Z|) \\
& = \langle \nabla_{\tau e_m}^{TM,\beta,\varepsilon} \nabla_{\tau e_k}^{TM,\beta,\varepsilon} f'_s, f'_t \rangle - \langle \nabla_{\tau e_k}^{TM,\beta,\varepsilon} \nabla_{\tau e_m}^{TM,\beta,\varepsilon} f'_s, f'_t \rangle - \left\langle \nabla_{[\tau e_m, \tau e_k]}^{TM,\beta,\varepsilon} f'_s, f'_t \right\rangle + O_{\beta,\varepsilon}(|Z|) \\
& = - \langle p \nabla_{\tau e_k}^{TM,\beta,\varepsilon} f'_s, \nabla_{\tau e_m}^{TM,\beta,\varepsilon} f'_t \rangle - \frac{1}{\beta^2 \varepsilon^2} \langle p_1^\perp \nabla_{\tau e_k}^{TM,\beta,\varepsilon} f'_s, \nabla_{\tau e_m}^{TM,\beta,\varepsilon} f'_t \rangle - \frac{1}{\beta^2} \langle p_2^\perp \nabla_{\tau e_k}^{TM,\beta,\varepsilon} f'_s, \nabla_{\tau e_m}^{TM,\beta,\varepsilon} f'_t \rangle \\
& + \langle p \nabla_{\tau e_m}^{TM,\beta,\varepsilon} f'_s, \nabla_{\tau e_k}^{TM,\beta,\varepsilon} f'_t \rangle + \frac{1}{\beta^2 \varepsilon^2} \langle p_1^\perp \nabla_{\tau e_m}^{TM,\beta,\varepsilon} f'_s, \nabla_{\tau e_k}^{TM,\beta,\varepsilon} f'_t \rangle + \frac{1}{\beta^2} \langle p_2^\perp \nabla_{\tau e_m}^{TM,\beta,\varepsilon} f'_s, \nabla_{\tau e_k}^{TM,\beta,\varepsilon} f'_t \rangle \\
& + \tau e_m (\langle \nabla_{\tau e_k}^{TM,\beta,\varepsilon} f'_s, f'_t \rangle) - \tau e_k (\langle \nabla_{\tau e_m}^{TM,\beta,\varepsilon} f'_s, f'_t \rangle) - \left\langle \nabla_{[\tau e_m, \tau e_k]}^{TM,\beta,\varepsilon} f'_s, f'_t \right\rangle + O_{\beta,\varepsilon}(|Z|) \\
& = O\left(\frac{\varepsilon^2}{\beta^2}\right) + O_{\beta,\varepsilon}(|Z|).
\end{aligned}$$

If  $1 \leq s \leq q, q+1 \leq t \leq q+q_1$ , one has

$$\begin{aligned}
(2.137) \quad & \frac{\varepsilon}{\beta} \langle R^{TM,\beta,\varepsilon}(\tau e_m, \tau e_k) \tau f_s, \tau f_t \rangle_{\beta,\varepsilon} = \frac{1}{\beta \varepsilon} \langle R^{TM,\beta,\varepsilon}(\tau e_m, \tau e_k) f'_s, f'_t \rangle + O_{\beta,\varepsilon}(|Z|) \\
& = \frac{1}{\beta \varepsilon} \langle \nabla_{\tau e_m}^{TM,\beta,\varepsilon} \nabla_{\tau e_k}^{TM,\beta,\varepsilon} f'_s, f'_t \rangle - \frac{1}{\beta \varepsilon} \langle \nabla_{\tau e_k}^{TM,\beta,\varepsilon} \nabla_{\tau e_m}^{TM,\beta,\varepsilon} f'_s, f'_t \rangle - \frac{1}{\beta \varepsilon} \left\langle \nabla_{[\tau e_m, \tau e_k]}^{TM,\beta,\varepsilon} f'_s, f'_t \right\rangle + O_{\beta,\varepsilon}(|Z|) \\
& = -\beta \varepsilon \langle p \nabla_{\tau e_k}^{TM,\beta,\varepsilon} f'_s, \nabla_{\tau e_m}^{TM,\beta,\varepsilon} f'_t \rangle - \frac{1}{\beta \varepsilon} \langle p_1^\perp \nabla_{\tau e_k}^{TM,\beta,\varepsilon} f'_s, \nabla_{\tau e_m}^{TM,\beta,\varepsilon} f'_t \rangle - \frac{\varepsilon}{\beta} \langle p_2^\perp \nabla_{\tau e_k}^{TM,\beta,\varepsilon} f'_s, \nabla_{\tau e_m}^{TM,\beta,\varepsilon} f'_t \rangle \\
& + \beta \varepsilon \langle p \nabla_{\tau e_m}^{TM,\beta,\varepsilon} f'_s, \nabla_{\tau e_k}^{TM,\beta,\varepsilon} f'_t \rangle + \frac{1}{\beta \varepsilon} \langle p_1^\perp \nabla_{\tau e_m}^{TM,\beta,\varepsilon} f'_s, \nabla_{\tau e_k}^{TM,\beta,\varepsilon} f'_t \rangle + \frac{\varepsilon}{\beta} \langle p_2^\perp \nabla_{\tau e_m}^{TM,\beta,\varepsilon} f'_s, \nabla_{\tau e_k}^{TM,\beta,\varepsilon} f'_t \rangle \\
& + \frac{1}{\beta \varepsilon} \tau e_m (\langle \nabla_{\tau e_k}^{TM,\beta,\varepsilon} f'_s, f'_t \rangle) - \frac{1}{\beta \varepsilon} \tau e_k (\langle \nabla_{\tau e_m}^{TM,\beta,\varepsilon} f'_s, f'_t \rangle) - \frac{1}{\beta \varepsilon} \left\langle \nabla_{[\tau e_m, \tau e_k]}^{TM,\beta,\varepsilon} f'_s, f'_t \right\rangle + O_{\beta,\varepsilon}(|Z|) \\
& = O\left(\frac{\varepsilon}{\beta}\right) + O_{\beta,\varepsilon}(|Z|).
\end{aligned}$$

If  $q+1 \leq s, t \leq q+q_1$ , one has, in view of (2.1),

$$\begin{aligned}
(2.138) \quad & \varepsilon^2 \langle R^{TM,\beta,\varepsilon}(\tau e_m, \tau e_k) \tau f_s, \tau f_t \rangle_{\beta,\varepsilon} = \varepsilon^2 \langle R^{TM,\beta,\varepsilon}(f'_s, f'_t) \tau e_m, \tau e_k \rangle + O_{\beta,\varepsilon}(|Z|) \\
& = \varepsilon^2 \left\langle \nabla_{f'_s}^{TM,\beta,\varepsilon} \nabla_{f'_t}^{TM,\beta,\varepsilon} \tau e_m, \tau e_k \right\rangle - \varepsilon^2 \left\langle \nabla_{f'_t}^{TM,\beta,\varepsilon} \nabla_{f'_s}^{TM,\beta,\varepsilon} \tau e_m, \tau e_k \right\rangle \\
& \quad - \varepsilon^2 \left\langle \nabla_{[f'_s, f'_t]}^{TM,\beta,\varepsilon} \tau e_m, \tau e_k \right\rangle + O_{\beta,\varepsilon}(|Z|) \\
& = -\varepsilon^2 \beta^2 \left\langle p \nabla_{f'_t}^{TM,\beta,\varepsilon} \tau e_m, \nabla_{f'_s}^{TM,\beta,\varepsilon} \tau e_k \right\rangle - \left\langle p_1^\perp \nabla_{f'_t}^{TM,\beta,\varepsilon} \tau e_m, \nabla_{f'_s}^{TM,\beta,\varepsilon} \tau e_k \right\rangle
\end{aligned}$$

$$\begin{aligned}
& -\varepsilon^2 \left\langle p_2^\perp \nabla_{f'_t}^{TM, \beta, \varepsilon} \tau e_m, \nabla_{f'_s}^{TM, \beta, \varepsilon} \tau e_k \right\rangle \\
& + \varepsilon^2 \beta^2 \left\langle p \nabla_{f'_s}^{TM, \beta, \varepsilon} \tau e_m, \nabla_{f'_t}^{TM, \beta, \varepsilon} \tau e_k \right\rangle + \left\langle p_1^\perp \nabla_{f'_s}^{TM, \beta, \varepsilon} \tau e_m, \nabla_{f'_t}^{TM, \beta, \varepsilon} \tau e_k \right\rangle \\
& + \varepsilon^2 \left\langle p_2^\perp \nabla_{f'_s}^{TM, \beta, \varepsilon} \tau e_m, \nabla_{f'_t}^{TM, \beta, \varepsilon} \tau e_k \right\rangle \\
& + \varepsilon^2 f'_s \left( \left\langle \nabla_{f'_t}^{TM, \beta, \varepsilon} \tau e_m, \tau e_k \right\rangle \right) - \varepsilon^2 f'_t \left( \left\langle \nabla_{f'_s}^{TM, \beta, \varepsilon} \tau e_m, \tau e_k \right\rangle \right) - \varepsilon^2 \left\langle \nabla_{[f'_s, f'_t]}^{TM, \beta, \varepsilon} \tau e_m, \tau e_k \right\rangle + O_{\beta, \varepsilon}(|Z|) \\
& = \left\langle p_1^\perp \nabla_{f'_t}^{TM, \beta, \varepsilon} \tau e_k, \nabla_{f'_s}^{TM, \beta, \varepsilon} \tau e_m \right\rangle - \left\langle \nabla_{f'_s}^{TM, \beta, \varepsilon} \tau e_k, p_1^\perp \nabla_{f'_t}^{TM, \beta, \varepsilon} \tau e_m \right\rangle + O\left(\frac{\varepsilon^2}{\beta^2}\right) + O_{\beta, \varepsilon}(|Z|).
\end{aligned}$$

From (2.138), one gets that for  $q+1 \leq s, t \leq q+q_1$ , one has

$$\begin{aligned}
(2.139) \quad & \left\langle R^{\mathcal{F}_1^\perp, \beta, \varepsilon}(\tau e_m, \tau e_k) f'_s, f'_t \right\rangle = \left\langle R^{TM, \beta, \varepsilon}(\tau e_m, \tau e_k) f'_s, f'_t \right\rangle \\
& + \beta^2 \varepsilon^2 \left\langle p \nabla_{\tau e_m}^{TM, \beta, \varepsilon} f'_s, \nabla_{\tau e_k}^{TM, \beta, \varepsilon} f'_t \right\rangle + \varepsilon^2 \left\langle p_2^\perp \nabla_{\tau e_m}^{TM, \beta, \varepsilon} f'_s, \nabla_{\tau e_k}^{TM, \beta, \varepsilon} f'_t \right\rangle \\
& - \beta^2 \varepsilon^2 \left\langle p \nabla_{\tau e_k}^{TM, \beta, \varepsilon} f'_s, \nabla_{\tau e_m}^{TM, \beta, \varepsilon} f'_t \right\rangle - \varepsilon^2 \left\langle p_2^\perp \nabla_{\tau e_k}^{TM, \beta, \varepsilon} f'_s, \nabla_{\tau e_m}^{TM, \beta, \varepsilon} f'_t \right\rangle \\
& = \left\langle p_1^\perp \nabla_{f'_t}^{TM, \beta, \varepsilon} \tau e_k, \nabla_{f'_s}^{TM, \beta, \varepsilon} \tau e_m \right\rangle - \left\langle \nabla_{f'_s}^{TM, \beta, \varepsilon} \tau e_k, p_1^\perp \nabla_{f'_t}^{TM, \beta, \varepsilon} \tau e_m \right\rangle + O\left(\frac{\varepsilon^2}{\beta^2}\right) + O_{\beta, \varepsilon}(|Z|).
\end{aligned}$$

From (2.23), (2.50), (2.55), (2.99), (2.101), (2.102), (2.120), (2.123), (2.125) and (2.134)-(2.139), one deduces that

$$\begin{aligned}
(2.140) \quad & \frac{1}{\beta^2} \sum_{i, j=1, i \neq j}^q \left\langle (1 - p_{T, \beta, \varepsilon}) \tau f_i(f_T) c_{\beta, \varepsilon}(\beta^{-1} \tau f_i) \tau \sigma, c_{\beta, \varepsilon}(\beta^{-1} \tau f_j) f_T \tilde{\nabla}_{\tau f_j}^{\mathcal{F}, \beta, \varepsilon}(\tau \sigma) \right\rangle \\
& + \frac{\varepsilon}{\beta} \sum_{i=1}^q \sum_{j=q+1}^{q+q_1} \left\langle (1 - p_{T, \beta, \varepsilon}) \tau f_i(f_T) c_{\beta, \varepsilon}(\beta^{-1} \tau f_i) \tau \sigma, c_{\beta, \varepsilon}(\varepsilon \tau f_j) f_T \tilde{\nabla}_{\tau f_j}^{\mathcal{F}, \beta, \varepsilon}(\tau \sigma) \right\rangle \\
& + \frac{\varepsilon}{\beta} \sum_{j=1}^q \sum_{i=q+1}^{q+q_1} \left\langle (1 - p_{T, \beta, \varepsilon}) \tau f_i(f_T) c_{\beta, \varepsilon}(\varepsilon \tau f_i) \tau \sigma, c_{\beta, \varepsilon}(\beta^{-1} \tau f_j) f_T \tilde{\nabla}_{\tau f_j}^{\mathcal{F}, \beta, \varepsilon}(\tau \sigma) \right\rangle \\
& + \varepsilon^2 \sum_{i, j=q+1, i \neq j}^{q+q_1} \left\langle (1 - p_{T, \beta, \varepsilon}) \tau f_i(f_T) c_{\beta, \varepsilon}(\varepsilon \tau f_i) \tau \sigma, (1 - p_{T, \beta, \varepsilon}) c_{\beta, \varepsilon}(\varepsilon \tau f_j) f_T \tilde{\nabla}_{\tau f_j}^{\mathcal{F}, \beta, \varepsilon}(\tau \sigma) \right\rangle \\
& = \frac{1}{\beta^2} \sum_{i=1}^q \sum_{t=q+1}^{q+q_1} \int_{s(M)} O\left(\left|p_1^\perp \nabla_{f'_t}^{TM, \beta, \varepsilon} \left(\nabla_{f'_i}^{\mathcal{F}_2^\perp} Z\right)\right|^2\right) |\sigma|^2 dv_{s(M)} \\
& + \frac{\varepsilon}{2\beta} \sum_{i=1}^q \sum_{t=q+1}^{q+q_1} \int_{s(M)} \left\langle c_{\beta, \varepsilon}(\beta^{-1} f_i) c_{\beta, \varepsilon}(\varepsilon f_t) \sigma, {}^Q \nabla_{f_t}^{\mathcal{F}, \phi_1(\mathcal{F}_1^\perp), \beta, \varepsilon} p_1^\perp \nabla_{f_t}^{TM, \beta, \varepsilon} \left(\nabla_{f_i}^{\mathcal{F}_2^\perp} Z\right) (\tau \sigma) \right\rangle dv_{s(M)} \\
& + O\left(\frac{\varepsilon}{\beta^4}\right) \int_{s(M)} |\sigma|^2 dv_{s(M)} + \sum_{k=1}^q O\left(\frac{\varepsilon}{\beta^3}\right) \int_{s(M)} |\sigma| \cdot \left| {}^Q \nabla_{f_k}^{\mathcal{F}, \phi_1(\mathcal{F}_1^\perp), \beta, \varepsilon}(\tau \sigma) \right| dv_{s(M)} \\
& + \sum_{k=q+1}^{q+q_1} O\left(\frac{\varepsilon^2}{\beta^2}\right) \int_{s(M)} |\sigma| \cdot \left| {}^Q \nabla_{f_k}^{\mathcal{F}, \phi_1(\mathcal{F}_1^\perp), \beta, \varepsilon}(\tau \sigma) \right| dv_{s(M)} \\
& + O\left(\frac{1}{\sqrt{T}}\right) \int_{s(M)} \left( |\sigma|^2 + \sum_{k=1}^{q+q_1} \left| {}^Q \nabla_{f_k}^{\mathcal{F}, \phi_1(\mathcal{F}_1^\perp), \beta, \varepsilon}(\tau \sigma) \right|^2 \right) dv_{s(M)}.
\end{aligned}$$

**2.8. A proof of Proposition 2.2.** By (2.65), (2.66), (2.73), (2.81), (2.83), (2.88) and (2.89), one has

$$\begin{aligned}
 (2.141) \quad & \frac{1}{\beta} \sum_{i=1}^q \sum_{j=1}^{q_2} \left\langle (1 - p_{T,\beta,\varepsilon}) c_{\beta,\varepsilon} (\beta^{-1} \tau f_i) \nabla_{\tau f_i}^{\mathcal{F}, \phi_1(\mathcal{F}_1^\perp), \beta, \varepsilon} J_{T,\beta,\varepsilon} \sigma, c_{\beta,\varepsilon} (\tau e_j) \nabla_{\tau e_j}^{\mathcal{F}, \phi_1(\mathcal{F}_1^\perp), \beta, \varepsilon} J_{T,\beta,\varepsilon} \sigma \right\rangle \\
 & + \varepsilon \sum_{i=q+1}^{q+q_2} \sum_{j=1}^{q_2} \left\langle (1 - p_{T,\beta,\varepsilon}) c_{\beta,\varepsilon} (\varepsilon \tau f_i) \nabla_{\tau f_i}^{\mathcal{F}, \phi_1(\mathcal{F}_1^\perp), \beta, \varepsilon} J_{T,\beta,\varepsilon} \sigma, c_{\beta,\varepsilon} (\tau e_j) \nabla_{\tau e_j}^{\mathcal{F}, \phi_1(\mathcal{F}_1^\perp), \beta, \varepsilon} J_{T,\beta,\varepsilon} \sigma \right\rangle \\
 & = O \left( 1 + \frac{\varepsilon^2}{\beta^2} + \frac{1}{\sqrt{T}} \right) \int_{s(M)} |\sigma|^2 dv_{s(M)} + O \left( \frac{1}{\sqrt{T}} \right) \sum_{i=1}^{q+q_1} \int_{s(M)} \left| Q \nabla_{f_i}^{\mathcal{F}, \phi_1(\mathcal{F}_1^\perp), \beta, \varepsilon} (\tau \sigma) \right|^2 dv_{s(M)}.
 \end{aligned}$$

On the other hand, by (2.98), one gets

$$\begin{aligned}
 (2.142) \quad & \sum_{i, k=1, i \neq k}^{q_2} \left\langle (1 - p_{T,\beta,\varepsilon}) c_{\beta,\varepsilon} (\tau e_i) \nabla_{\tau e_i}^{\mathcal{F}, \phi_1(\mathcal{F}_1^\perp), \beta, \varepsilon} J_{T,\beta,\varepsilon} \sigma, c_{\beta,\varepsilon} (\tau e_k) \nabla_{\tau e_k}^{\mathcal{F}, \phi_1(\mathcal{F}_1^\perp), \beta, \varepsilon} J_{T,\beta,\varepsilon} \sigma \right\rangle \\
 & = O \left( \varepsilon^2 + \frac{1}{\sqrt{T}} \right) \int_{s(M)} |\sigma|^2 dv_{s(M)} + O \left( \frac{1}{\sqrt{T}} \right) \sum_{i=1}^{q+q_1} \int_{s(M)} \left| Q \nabla_{f_i}^{\mathcal{F}, \phi_1(\mathcal{F}_1^\perp), \beta, \varepsilon} (\tau \sigma) \right|^2 dv_{s(M)}.
 \end{aligned}$$

From (2.8), (2.9), (2.21), (2.22), (2.32), (2.47), (2.54), (2.60), (2.63), (2.64) and (2.140)-(2.142), one deduces that

$$\begin{aligned}
 (2.143) \quad & \left\| p_{T,\beta,\varepsilon} D^{\mathcal{F}, \phi_1(\mathcal{F}_1^\perp), \beta, \varepsilon} J_{T,\beta,\varepsilon} \sigma \right\|_0^2 \geq \left\langle \left( \frac{k^{\mathcal{F}}}{4\beta^2} + O \left( \frac{1}{\beta} + \frac{\varepsilon^2}{\beta^2} \right) \right) J_{T,\beta,\varepsilon} \sigma, J_{T,\beta,\varepsilon} \sigma \right\rangle \\
 & + \frac{1}{\beta^2} \sum_{i=1}^q \left\| p_{T,\beta,\varepsilon} \nabla_{\tau f_i}^{\mathcal{F}, \phi_1(\mathcal{F}_1^\perp), \beta, \varepsilon} J_{T,\beta,\varepsilon} \sigma \right\|_0^2 + \varepsilon^2 \sum_{i=q+1}^{q+q_1} \left\| p_{T,\beta,\varepsilon} \nabla_{\tau f_i}^{\mathcal{F}, \phi_1(\mathcal{F}_1^\perp), \beta, \varepsilon} J_{T,\beta,\varepsilon} \sigma \right\|_0^2 \\
 & + \sum_{i=1}^{q_2} \left\| \nabla_{\tau e_i}^{\mathcal{F}, \phi_1(\mathcal{F}_1^\perp), \beta, \varepsilon} J_{T,\beta,\varepsilon} \sigma \right\|_0^2 - \sum_{i=1}^{q_2} \left\| (1 - p_{T,\beta,\varepsilon}) c_{\beta,\varepsilon} (\tau e_i) \nabla_{\tau e_i}^{\mathcal{F}, \phi_1(\mathcal{F}_1^\perp), \beta, \varepsilon} J_{T,\beta,\varepsilon} \sigma \right\|_0^2 \\
 & + \frac{1}{\beta^2} \sum_{i=1}^q \sum_{t=q+1}^{q+q_1} \int_{s(M)} O \left( \left| p_1^\perp \nabla_{f_t}^{T\mathcal{M}, \beta, \varepsilon} \left( \nabla_{f_i}^{\mathcal{F}_2^\perp} Z \right) \right|^2 \right) |\sigma|^2 dv_{s(M)} \\
 & - \frac{\varepsilon}{\beta} \sum_{i=1}^q \sum_{t=q+1}^{q+q_1} \int_{s(M)} \left\langle c_{\beta,\varepsilon} (\beta^{-1} f_i) c_{\beta,\varepsilon} (\varepsilon f_t) \sigma, {}^Q \nabla_{p_1^\perp \nabla_{f_t}^{T\mathcal{M}, \beta, \varepsilon} \left( \nabla_{f_i}^{\mathcal{F}_2^\perp} Z \right)}^{\mathcal{F}, \phi_1(\mathcal{F}_1^\perp), \beta, \varepsilon} (\tau \sigma) \right\rangle dv_{s(M)} \\
 & + O \left( 1 + \frac{\varepsilon}{\beta^4} \right) \int_{s(M)} |\sigma|^2 dv_{s(M)} + \sum_{k=1}^q O \left( \frac{\varepsilon}{\beta^3} \right) \int_{s(M)} |\sigma| \cdot \left| Q \nabla_{f_k}^{\mathcal{F}, \phi_1(\mathcal{F}_1^\perp), \beta, \varepsilon} (\tau \sigma) \right| dv_{s(M)} \\
 & + \sum_{k=q+1}^{q+q_1} O \left( \frac{\varepsilon^2}{\beta^2} \right) \int_{s(M)} |\sigma| \cdot \left| Q \nabla_{f_k}^{\mathcal{F}, \phi_1(\mathcal{F}_1^\perp), \beta, \varepsilon} (\tau \sigma) \right| dv_{s(M)} \\
 & + O \left( \frac{1}{\sqrt{T}} \right) \int_{s(M)} \left( |\sigma|^2 + \sum_{k=1}^{q+q_1} \left| Q \nabla_{f_k}^{\mathcal{F}, \phi_1(\mathcal{F}_1^\perp), \beta, \varepsilon} (\tau \sigma) \right|^2 \right) dv_{s(M)}.
 \end{aligned}$$

Clearly, for any  $1 \leq i \leq q_2$ , one has

$$(2.144) \quad \left\| \nabla_{\tau e_i}^{\mathcal{F}, \phi_1(\mathcal{F}_1^\perp), \beta, \varepsilon} J_{T, \varepsilon} \sigma \right\|_0^2 - \left\| (1 - p_{T, \beta, \varepsilon}) c_{\beta, \varepsilon}(\tau e_i) \nabla_{\tau e_i}^{\mathcal{F}, \phi_1(\mathcal{F}_1^\perp), \beta, \varepsilon} J_{T, \beta, \varepsilon} \sigma \right\|_0^2 \\ \geq \left\| \nabla_{\tau e_i}^{\mathcal{F}, \phi_1(\mathcal{F}_1^\perp), \beta, \varepsilon} J_{T, \beta, \varepsilon} \sigma \right\|_0^2 - \left\| c_{\beta, \varepsilon}(\tau e_i) \nabla_{\tau e_i}^{\mathcal{F}, \phi_1(\mathcal{F}_1^\perp), \beta, \varepsilon} J_{T, \beta, \varepsilon} \sigma \right\|_0^2 = 0.$$

Also recall that we have assumed that  $k^{\mathcal{F}} \geq \eta > 0$  over  $U_\alpha(\mathcal{F}_2^\perp)$ . Then one has

$$(2.145) \quad \left\langle \left( \frac{k^{\mathcal{F}}}{4\beta^2} + O\left(\frac{1}{\beta} + \frac{\varepsilon^2}{\beta^2}\right) \right) J_{T, \beta, \varepsilon} \sigma, J_{T, \beta, \varepsilon} \sigma \right\rangle \geq \int_{s(M)} \left( \frac{\eta}{4\beta^2} + O\left(\frac{1}{\beta} + \frac{\varepsilon^2}{\beta^2}\right) \right) |\sigma|^2 dv_{s(M)}.$$

For  $1 \leq i \leq q + q_1$ , by (2.17) and (2.23)-(2.25), one has,

$$(2.146) \quad p_{T, \beta, \varepsilon} \nabla_{\tau f_i}^{\mathcal{F}, \phi_1(\mathcal{F}_1^\perp), \beta, \varepsilon} J_{T, \beta, \varepsilon} \sigma = p_{T, \beta, \varepsilon} \left( \tau f_i(f_T) \tau \sigma + f_T \nabla_{\tau f_i}^{\mathcal{F}, \phi_1(\mathcal{F}_1^\perp), \beta, \varepsilon} (\tau \sigma) \right) \\ = \left( \int_{\mathcal{M}_x} f_T \tau f_i(f_T) k dv_{\mathcal{M}_x} \right) J_{T, \beta, \varepsilon} \sigma + p_{T, \beta, \varepsilon} \left( f_T Q \nabla_{\tau f_i}^{\mathcal{F}, \phi_1(\mathcal{F}_1^\perp), \beta, \varepsilon} (\tau \sigma) \right).$$

From (2.26) and Lemma 2.7, one deduces that the following formula holds for any  $1 \leq i \leq q + q_1$ ,

$$(2.147) \quad \left\| p_{T, \beta, \varepsilon} \left( f_T Q \nabla_{\tau f_i}^{\mathcal{F}, \phi_1(\mathcal{F}_1^\perp), \beta, \varepsilon} (\tau \sigma) \right) \right\|_0^2 = \int_{s(M)} \left| Q \nabla_{f_i}^{\mathcal{F}, \phi_1(\mathcal{F}_1^\perp), \beta, \varepsilon} (\tau \sigma) \right|^2 dv_{s(M)} \\ + O\left(\frac{1}{\sqrt{T}}\right) \int_{s(M)} |\sigma|^2 dv_{s(M)} + O\left(\frac{1}{\sqrt{T}}\right) \sum_{j=1}^{q+q_1} \int_{s(M)} \left| Q \nabla_{f_j}^{\mathcal{F}, \phi_1(\mathcal{F}_1^\perp), \beta, \varepsilon} (\tau \sigma) \right|^2 dv_{s(M)}.$$

If  $1 \leq i \leq q$ , by (2.34) and (2.50), one gets

$$(2.148) \quad \int_{\mathcal{M}_x} f_T \tau f_i(f_T) k dv_{\mathcal{M}_x} = O\left(1 + \frac{1}{\sqrt{T}}\right).$$

If  $q + 1 \leq i \leq q + q_1$ , by (2.35) and (2.50), one gets

$$(2.149) \quad \int_{\mathcal{M}_x} f_T \tau f_i(f_T) k dv_{\mathcal{M}_x} = O\left(\frac{1}{\beta^2} + \frac{1}{\sqrt{T}}\right).$$

Recall the following obvious inequality,

$$(2.150) \quad |a + b|^2 \geq \frac{|a|^2}{2} - |b|^2.$$

By (2.146)-(2.150), one gets that for  $0 < \delta \leq 1$  sufficiently small,

$$\begin{aligned}
(2.151) \quad & \frac{1}{\beta^2} \sum_{i=1}^q \left\| p_{T,\beta,\varepsilon} \nabla_{\tau f_i}^{\mathcal{F},\phi_1(\mathcal{F}_1^\perp),\beta,\varepsilon} J_{T,\beta,\varepsilon} \sigma \right\|_0^2 + \varepsilon^2 \sum_{i=q+1}^{q+q_1} \left\| p_{T,\beta,\varepsilon} \nabla_{\tau f_i}^{\mathcal{F},\phi_1(\mathcal{F}_1^\perp),\beta,\varepsilon} J_{T,\beta,\varepsilon} \sigma \right\|_0^2 \\
& \geq \sum_{i=1}^q \frac{\varepsilon^\delta}{\beta^2} \left\| p_{T,\beta,\varepsilon} \nabla_{\tau f_i}^{\mathcal{F},\phi_1(\mathcal{F}_1^\perp),\beta,\varepsilon} J_{T,\beta,\varepsilon} \sigma \right\|_0^2 + \varepsilon^2 \sum_{i=q+1}^{q+q_1} \left\| p_{T,\beta,\varepsilon} \nabla_{\tau f_i}^{\mathcal{F},\phi_1(\mathcal{F}_1^\perp),\beta,\varepsilon} J_{T,\beta,\varepsilon} \sigma \right\|_0^2 \\
& \geq \int_{s(M)} O\left(\frac{\varepsilon^\delta}{\beta^2} + \frac{\varepsilon^2}{\beta^4} + \frac{1}{\sqrt{T}}\right) |\sigma|^2 dv_{s(M)} + \frac{\varepsilon^\delta}{4\beta^2} \sum_{i=1}^q \int_{s(M)} \left| Q \nabla_{f_i}^{\mathcal{F},\phi_1(\mathcal{F}_1^\perp),\beta,\varepsilon} (\tau \sigma) \right|^2 dv_{s(M)} \\
& + \frac{\varepsilon^2}{4} \sum_{i=q+1}^{q+q_1} \int_{s(M)} \left| Q \nabla_{f_i}^{\mathcal{F},\phi_1(\mathcal{F}_1^\perp),\beta,\varepsilon} (\tau \sigma) \right|^2 dv_{s(M)} + O\left(\frac{1}{\sqrt{T}}\right) \sum_{i=1}^{q+q_1} \int_{s(M)} \left| Q \nabla_{f_i}^{\mathcal{F},\phi_1(\mathcal{F}_1^\perp),\beta,\varepsilon} (\tau \sigma) \right|^2 dv_{s(M)}.
\end{aligned}$$

From (2.143)-(2.145) and (2.151), one deduces that

$$\begin{aligned}
(2.152) \quad & \left\| p_{T,\beta,\varepsilon} D^{\mathcal{F},\phi_1(\mathcal{F}_1^\perp),\beta,\varepsilon} J_{T,\beta,\varepsilon} \sigma \right\|_0^2 \geq \left( \frac{\eta}{4\beta^2} + O\left(\frac{1}{\beta} + \frac{\varepsilon^\delta}{\beta^2} + \frac{\varepsilon}{\beta^4}\right) \right) \int_{s(M)} |\sigma|^2 dv_{s(M)} \\
& + \frac{1}{\beta^2} \sum_{i=1}^q \sum_{t=q+1}^{q+q_1} \int_{s(M)} O\left(\left| p_1^\perp \nabla_{f_t}^{T\mathcal{M},\beta,\varepsilon} \left( \nabla_{f_i}^{\mathcal{F}_2^\perp} Z \right) \right|^2\right) |\sigma|^2 dv_{s(M)} \\
& - \frac{\varepsilon}{\beta} \sum_{i=1}^q \sum_{t=q+1}^{q+q_1} \int_{s(M)} \left\langle c_{\beta,\varepsilon}(\beta^{-1} f_i) c_{\beta,\varepsilon}(\varepsilon f_t) \sigma, Q \nabla_{p_1^\perp \nabla_{f_t}^{T\mathcal{M},\beta,\varepsilon} \left( \nabla_{f_i}^{\mathcal{F}_2^\perp} Z \right)}^{\mathcal{F},\phi_1(\mathcal{F}_1^\perp),\beta,\varepsilon} (\tau \sigma) \right\rangle dv_{s(M)} \\
& + \left( \frac{\varepsilon^\delta}{4\beta^2} + O\left(\frac{\varepsilon}{\beta^2}\right) \right) \sum_{k=1}^q \int_{s(M)} \left| Q \nabla_{f_k}^{\mathcal{F},\phi_1(\mathcal{F}_1^\perp),\beta,\varepsilon} (\tau \sigma) \right|^2 dv_{s(M)} \\
& + \left( \frac{\varepsilon^2}{4} + O(\varepsilon^3) \right) \sum_{k=q+1}^{q+q_1} \int_{s(M)} \left| Q \nabla_{f_k}^{\mathcal{F},\phi_1(\mathcal{F}_1^\perp),\beta,\varepsilon} (\tau \sigma) \right|^2 dv_{s(M)} \\
& + O\left(\frac{1}{\sqrt{T}}\right) \int_{s(M)} \left( |\sigma|^2 + \sum_{k=1}^{q+q_1} \left| Q \nabla_{f_k}^{\mathcal{F},\phi_1(\mathcal{F}_1^\perp),\beta,\varepsilon} (\tau \sigma) \right|^2 \right) dv_{s(M)}.
\end{aligned}$$

From (2.152), one gets (2.20).

The proof of Proposition 2.2 is completed.

**2.9. A quasi-positivity result on a doubled Connes fibration.** In this subsection, we will apply Proposition 2.2 to a specific Connes type fibration over  $(M, F)$  so that one can further estimate the second term in the right hand side of (2.20) into a more suitable form.

We start by recalling the original Connes fibration over  $(M, F)$  constructed in [6].

For any oriented vector space  $E$  of rank  $n$ , let  $\mathcal{E}$  be the set of all Euclidean metrics on  $E$ . It is well known that  $\mathcal{E}$  is the homogeneous space  $GL(n, \mathbf{R})^+/SO(n)$  (with  $\dim \mathcal{E} = \frac{1}{2}(\text{rk}(E) + 1)\text{rk}(E)$ ), which carries a natural Riemannian metric of nonpositive sectional curvature (cf. [11]). In particular, any two points of  $\mathcal{E}$  can be joined by a unique geodesic.

Following [6, Section 5], let  $\pi : \mathcal{M} \rightarrow M$  be the fibration over  $M$  such that for any  $x \in M$ ,  $\mathcal{M}_x = \pi^{-1}(x)$  is the space of Euclidean metrics on the linear space  $T_x M / F_x$ . Clearly,  $\mathcal{M}$  is noncompact.

Let  $T^V \mathcal{M}$  denote the vertical tangent bundle of the fibration  $\pi : \mathcal{M} \rightarrow M$ . Then it carries a natural metric  $g^{T^V \mathcal{M}}$  such that any two points  $p, q \in \mathcal{M}_x$ , with  $x \in M$ , can be joined by a unique geodesic in  $\mathcal{M}_x$ .

By using the Bott connection [5] on  $TM/F$ , one can lift  $F$  to an integrable subbundle  $\mathcal{F}$  of  $T\mathcal{M}$ .<sup>11</sup> Moreover,  $\mathcal{F}$  is spin and carries a spin structure induced from that of  $F$ .

Let  $g^F$  be a Euclidean metric on  $F$ . Then it lifts to a Euclidean metric  $g^{\mathcal{F}} = \pi^* g^F$  on  $\mathcal{F}$ . Recall that we have assumed that  $g^F$  is of positive (leafwise) scalar curvature. Then  $g^{\mathcal{F}}$  is also of positive (leafwise) scalar curvature.

For any  $v \in \mathcal{M}$ ,  $T_v \mathcal{M} / (\mathcal{F}_v \oplus T_v^V \mathcal{M})$  identifies with  $T_{\pi(v)} M / F_{\pi(v)}$  under the projection  $\pi : \mathcal{M} \rightarrow M$ . By definition,  $v$  determines a metric on  $T_{\pi(v)} M / F_{\pi(v)}$ , thus it also determines a metric on  $T_v \mathcal{M} / (\mathcal{F}_v \oplus T_v^V \mathcal{M})$ . In this way,  $T\mathcal{M} / (\mathcal{F} \oplus T^V \mathcal{M})$  carries a canonically induced metric.

Let  $\mathcal{F}_1^\perp$  be a subbundle of  $T\mathcal{M}$  such that we have a splitting  $T\mathcal{M} = (\mathcal{F} \oplus T^V \mathcal{M}) \oplus \mathcal{F}_1^\perp$ . Then  $\mathcal{F}_1^\perp$  can be identified with  $T\mathcal{M} / (\mathcal{F} \oplus T^V \mathcal{M})$  and carries a canonically induced metric  $g^{\mathcal{F}_1^\perp}$ . Also, as before we denote  $\mathcal{F}_2^\perp = T^V \mathcal{M}$ .

By [6, Lemma 5.2],  $(\mathcal{M}, \mathcal{F})$  admits an almost isometric structure in the sense of Definition 1.1, with the metrics given in (1.4) and/or (2.2). In particular, (1.5) holds.<sup>12</sup>

Take a metric on  $TM/F$ . This is equivalent to taking an embedded section  $s : M \hookrightarrow \mathcal{M}$  of the Connes fibration  $\pi : \mathcal{M} \rightarrow M$ . Then we have a canonical inclusion  $s(M) \subset \mathcal{M}$ , as well as an induced fibration  $s \circ \pi : \mathcal{M} \rightarrow s(M)$ .

Now by construction, we know that, as we have indicated in Section 2.1,  $\mathcal{F} \oplus \mathcal{F}_2^\perp$  is an integrable subbundle of  $T\mathcal{M}$  with  $T\mathcal{M} / (\mathcal{F} \oplus \mathcal{F}_2^\perp) \simeq \mathcal{F}_1^\perp$ . Moreover,  $\mathcal{F}_1^\perp$  carries a canonically determined metric.

Let  $\widehat{\pi} : \widehat{\mathcal{M}} \rightarrow (\mathcal{M}, \mathcal{F} \oplus \mathcal{F}_2^\perp)$  be the Connes fibration over  $\mathcal{M}$  obtained by taking fibers as the spaces of Euclidean metrics on (fiberwise)  $\mathcal{F}_1^\perp$ . Then the canonical metric on  $\mathcal{F}_1^\perp$  determines a canonical embedded section  $\widehat{s} : \mathcal{M} \hookrightarrow \widehat{\mathcal{M}}$ .

By composition, we get a fibration  $\widehat{\pi} = \pi \circ \widehat{\pi} : \widehat{\mathcal{M}} \rightarrow M$ , as well as an embedded section  $\widehat{s} = \widehat{s} \circ s : M \hookrightarrow \widehat{\mathcal{M}}$ .

Let  $\widehat{\mathcal{F}} \oplus \widehat{\mathcal{F}}_{21}^\perp = \widehat{\pi}^*(\mathcal{F} \oplus \mathcal{F}_2^\perp)$  be the integrable subbundle of  $T\widehat{\mathcal{M}}$  so that

$$(2.153) \quad \widehat{\mathcal{F}} = \widehat{\pi}^* \mathcal{F} = \widetilde{\pi}^* F, \quad \widehat{\mathcal{F}}_{21}^\perp = \widehat{\pi}^* \mathcal{F}_2^\perp.$$

<sup>11</sup>Indeed, the Bott connection on  $TM/F$  determines an integrable lift  $\widetilde{\mathcal{F}}$  of  $F$  in  $T\widetilde{\mathcal{M}}$ , where  $\widetilde{\mathcal{M}} = GL(TM/F)^+$  is the  $GL(q_1, \mathbf{R})^+$  (with  $q_1 = \text{rk}(TM/F)$ ) principal bundle of oriented frames over  $M$ . Now as  $\widetilde{\mathcal{M}}$  is a principal  $SO(q_1)$  bundle over  $\mathcal{M}$ ,  $\widetilde{\mathcal{F}}$  determines an integrable subbundle  $\mathcal{F}$  of  $T\mathcal{M}$ .

<sup>12</sup>In fact, for any  $X \in \Gamma(F)$ , let  $\mathcal{X} \in \Gamma(\mathcal{F})$  denote the lift of  $X$ . Let  $\varphi_t$  (with  $t$  close to zero) be the one parameter family of diffeomorphisms on  $\mathcal{M}$  generated by  $\mathcal{X}$ . Then each  $\varphi_t$  acts on the complete transversal to  $\mathcal{F}$  in  $\mathcal{M}$ . The differential of  $\varphi_t$ , when acting on the complete transversal, maps each  $(\mathcal{F}_1^\perp + \mathcal{F}_2^\perp)_x$  ( $x \in \mathcal{M}$ ) to  $(\mathcal{F}_1^\perp + \mathcal{F}_2^\perp)_{\varphi_t(x)}$  and verifies [6, Lemma 5.2]. By taking derivative at  $t = 0$ , one gets (1.5).

Since  $\mathcal{F}$ ,  $\mathcal{F}_2^\perp$  are integrable subbundles of  $T\mathcal{M}$ , one sees that  $\widehat{\mathcal{F}}$ ,  $\widehat{\mathcal{F}}_{21}^\perp$  are integrable subbundles of  $T\widehat{\mathcal{M}}$ . They carry canonically lifted metrics  $g^{\widehat{\mathcal{F}}} = \widehat{\pi}^*g^{\mathcal{F}} = \widetilde{\pi}^*g^{\mathcal{F}}$  and  $g^{\widehat{\mathcal{F}}_{21}^\perp} = \widehat{\pi}^*g^{\mathcal{F}_2^\perp}$  respectively.

Let  $\widehat{\mathcal{F}}_{22}$  be the vertical tangent bundle of the Connes fibration  $\widehat{\pi} : \widehat{\mathcal{M}} \rightarrow \mathcal{M}$ . Then it carries a canonical metric  $g^{\widehat{\mathcal{F}}_{22}}$ .

Set

$$(2.154) \quad \widehat{\mathcal{F}}_2^\perp = \widehat{\mathcal{F}}_{21}^\perp \oplus \widehat{\mathcal{F}}_{22}^\perp, \quad \text{with} \quad g^{\widehat{\mathcal{F}}_2^\perp} = g^{\widehat{\mathcal{F}}_{21}^\perp} \oplus g^{\widehat{\mathcal{F}}_{22}^\perp}.$$

Let  $\widehat{\mathcal{F}}_1^\perp \simeq T\widehat{\mathcal{M}}/(\widehat{\mathcal{F}} \oplus \widehat{\mathcal{F}}_2^\perp)$  be a subbundle of  $T\widehat{\mathcal{M}}$  transverse to  $\widehat{\mathcal{F}} \oplus \widehat{\mathcal{F}}_2^\perp$ . Then it carries a canonically determined metric  $g^{\widehat{\mathcal{F}}_1^\perp}$ . Without loss of generality, we assume that

$$(2.155) \quad \widehat{\mathcal{F}}_1^\perp \Big|_{\widehat{s}(\mathcal{M})} = \widehat{s}_*\mathcal{F}_1^\perp \subset T\widehat{s}(\mathcal{M}), \quad g^{\widehat{\mathcal{F}}_1^\perp} \Big|_{\widehat{s}(\mathcal{M})} = \widehat{s}_*g^{\mathcal{F}_1^\perp}.$$

Clearly, we can make the orthogonal splitting

$$(2.156) \quad T\widehat{\mathcal{M}} = \widehat{\mathcal{F}} \oplus \widehat{\mathcal{F}}_1^\perp \oplus \widehat{\mathcal{F}}_2^\perp, \quad g^{T\widehat{\mathcal{M}}} = g^{\widehat{\mathcal{F}}} \oplus g^{\widehat{\mathcal{F}}_1^\perp} \oplus g^{\widehat{\mathcal{F}}_2^\perp}.$$

We claim that  $\widetilde{\pi} : (\widehat{\mathcal{M}}, \widehat{\mathcal{F}}) \rightarrow (M, F)$  is a Connes type fibration in the sense of Definition 2.1.

Indeed, it is clear that for  $X \in \Gamma(\widehat{\mathcal{F}})$ ,  $U, V \in \Gamma(\widehat{\mathcal{F}}_{2i}^\perp)$  ( $i = 1$  or  $2$ ) and for  $U, V \in \Gamma(\widehat{\mathcal{F}}_1^\perp)$ , the first equality of (1.5) holds. If  $X \in \Gamma(\widehat{\mathcal{F}})$ ,  $U \in \Gamma(\widehat{\mathcal{F}}_{21}^\perp)$  and  $V \in \Gamma(\widehat{\mathcal{F}}_{22}^\perp)$ , then one has

$$(2.157) \quad \langle [X, U], V \rangle = \langle [X, V], U \rangle = 0.$$

In fact, the first one clearly vanishes, while for the second one, if  $X$  is lifted from  $\mathcal{F}$ , then  $[X, V] \in \Gamma(\widehat{\mathcal{F}}_{22}^\perp)$  so that (2.157) holds. On the other hand, for any function  $f$  on  $\widehat{\mathcal{M}}$ , one has  $[fX, V] = f[X, V] - V(f)X$ , from which one sees that (2.157) holds in general. Thus the first formula in (1.5) holds for  $(\widehat{\mathcal{M}}, \widehat{\mathcal{F}})$ . One sees also that the second identity in (1.5) holds for  $(\widehat{\mathcal{M}}, \widehat{\mathcal{F}})$ .

Clearly, the other conditions in Definition 2.1 also hold. Thus,  $(\widehat{\mathcal{M}}, \widehat{\mathcal{F}})$  is a Connes type fibration over  $(M, F)$  in the sense of Definition 2.1. In particular, (2.20) holds for the embedding  $\widetilde{s} : M \hookrightarrow \widehat{\mathcal{M}}$  under the corresponding rescaling involving  $\beta$  and  $\varepsilon$ .

Let us rewrite (2.20) in the current situation, which now holds on  $\widetilde{s}(M) = \widehat{s}(s(M)) \subset \widehat{s}(\mathcal{M}) \subset \widehat{\mathcal{M}}$ , as

$$(2.158) \quad \begin{aligned} \left\| p_{T, \beta, \varepsilon} D^{\widehat{\mathcal{F}}, \phi_1(\widehat{\mathcal{F}}_1^\perp), \beta, \varepsilon} J_{T, \beta, \varepsilon} \sigma \right\|_0^2 &\geq \left( \frac{\eta}{4\beta^2} - C_1 \left( \frac{1}{\beta} + \frac{\varepsilon^\delta}{\beta^2} + \frac{\varepsilon}{\beta^4} \right) \right) \int_{\widetilde{s}(M)} |\sigma|^2 dv_{\widetilde{s}(M)} \\ &\quad - \frac{C_2}{\beta^2} \sum_{i=1}^q \sum_{t=q+1}^{q+q_1} \int_{\widetilde{s}(M)} \left| p_1^\perp \nabla_{\widehat{f}_t}^{T\widehat{\mathcal{M}}, \beta, \varepsilon} \left( \nabla_{\widehat{f}_i}^{\widehat{\mathcal{F}}_2^\perp} Z \right) \right|^2 \cdot |\sigma|^2 dv_{\widetilde{s}(M)} \\ &\quad + \frac{\varepsilon^\delta}{8\beta^2} \sum_{k=1}^q \int_{\widetilde{s}(M)} \left| Q \nabla_{\widehat{f}_k}^{\widehat{\mathcal{F}}, \phi_1(\widehat{\mathcal{F}}_1^\perp), \beta, \varepsilon} (\tau \sigma) \right|^2 dv_{\widetilde{s}(M)} + \frac{\varepsilon^2}{8} \sum_{k=q+1}^{q+q_1} \int_{\widetilde{s}(M)} \left| Q \nabla_{\widehat{f}_k}^{\widehat{\mathcal{F}}, \phi_1(\widehat{\mathcal{F}}_1^\perp), \beta, \varepsilon} (\tau \sigma) \right|^2 dv_{\widetilde{s}(M)} \\ &\quad - \frac{C_{\beta, \varepsilon}}{\sqrt{T}} \int_{\widetilde{s}(M)} \left( |\sigma|^2 + \sum_{k=1}^{q+q_1} \left| Q \nabla_{\widehat{f}_k}^{\widehat{\mathcal{F}}, \phi_1(\widehat{\mathcal{F}}_1^\perp), \beta, \varepsilon} (\tau \sigma) \right|^2 \right) dv_{\widetilde{s}(M)}, \end{aligned}$$

where the integrations in the right hand side are now taken on  $\widetilde{s}(M)$ .

Now  $Z = Z_1 + Z_2$  with  $Z_1 \in \Gamma(\widehat{\mathcal{F}}_{21})$  and  $Z_2 \in \Gamma(\widehat{\mathcal{F}}_{22})$ . By the property (1.5) of the Connes fibration  $(\widehat{\mathcal{M}}, \widehat{\mathcal{F}} \oplus \widehat{\mathcal{F}}_{21}^\perp)$  over  $(\mathcal{M}, \mathcal{F} \oplus \mathcal{F}_2^\perp)$ , one sees that for any  $1 \leq i \leq q$ ,  $q+1 \leq t \leq q+q_1$ , the following identity holds on  $\widetilde{s}(M)$ ,

$$(2.159) \quad p_1^\perp \nabla_{\widehat{f}_t}^{T\widehat{\mathcal{M}}, \beta, \varepsilon} \left( \nabla_{\widehat{f}_i}^{\widehat{\mathcal{F}}_2^\perp} Z_1 \right) = O(\varepsilon^2).$$

On the other hand, for  $1 \leq i \leq q$ , without loss of generality, let  $f_i \in \Gamma(\mathcal{F})$  on  $\mathcal{M}$  be such that  $\widehat{f}_i = \widehat{\pi}^* f_i$ , then one has  $\widehat{f}_i - \widehat{s}_* f_i \in \Gamma(\widehat{\mathcal{F}}_{22})$  from which the following identity holds on  $\widehat{s}(\mathcal{M})$ :

$$(2.160) \quad \widehat{f}_i = \widehat{s}_* f_i + \nabla_{\widehat{f}_i - \widehat{s}_* f_i}^{\widehat{\mathcal{F}}_2^\perp} Z_2 = \widehat{s}_* f_i + \nabla_{\widehat{f}_i}^{\widehat{\mathcal{F}}_2^\perp} Z_2,$$

where the second equality comes from the fact that  $Z_2$  is identically zero on  $\widehat{s}(\mathcal{M})$ .

By (1.5) for the Connes fibration  $(\widehat{\mathcal{M}}, \widehat{\mathcal{F}} \oplus \widehat{\mathcal{F}}_{21}^\perp)$ , one finds that for  $q+1 \leq t \leq q+q_1$ ,

$$(2.161) \quad p_1^\perp \nabla_{\widehat{f}_t}^{T\widehat{\mathcal{M}}, \beta, \varepsilon} \widehat{f}_i = O(\beta^2 \varepsilon^2).$$

On the other hand, by (1.5) for the Connes fibration  $(\widehat{s}(\mathcal{M}), \widehat{s}_* \mathcal{F}) \simeq (\mathcal{M}, \mathcal{F})$  (cf. (1.12)), (2.155) and (2.160), one finds that the following identities hold on  $\widehat{s}(\mathcal{M})$ ,

$$(2.162) \quad \begin{aligned} p_1^\perp \nabla_{\widehat{f}_t}^{T\widehat{\mathcal{M}}, \beta, \varepsilon} (\widehat{s}_* f_i) &= -\frac{\varepsilon^2}{2} \sum_{s=q+1}^{q+q_1} \left\langle \widehat{s}_* f_i, [\widehat{f}_t, \widehat{f}_s] \right\rangle_{\beta, \varepsilon} \widehat{f}_s \\ &= -\frac{\beta^2 \varepsilon^2}{2} \sum_{s=q+1}^{q+q_1} \left\langle \widehat{f}_i, [\widehat{f}_t, \widehat{f}_s] \right\rangle \widehat{f}_s + \frac{\varepsilon^2}{2} \sum_{s=q+1}^{q+q_1} \left\langle \nabla_{\widehat{f}_i}^{\widehat{\mathcal{F}}_2^\perp} Z_2, [\widehat{f}_t, \widehat{f}_s] \right\rangle \widehat{f}_s = O(\varepsilon^2). \end{aligned}$$

From (2.159)-(2.162), one finds that the following identity holds on  $\widetilde{s}(M)$ ,

$$(2.163) \quad p_1^\perp \nabla_{\widehat{f}_t}^{T\widehat{\mathcal{M}}, \beta, \varepsilon} \left( \nabla_{\widehat{f}_i}^{\widehat{\mathcal{F}}_2^\perp} Z \right) = O(\varepsilon^2).$$

By (2.163), one can refine (2.158) to the following formula,

$$(2.164) \quad \begin{aligned} \left\| p_{T, \beta, \varepsilon} D^{\widehat{\mathcal{F}}, \phi_1(\widehat{\mathcal{F}}_1^\perp), \beta, \varepsilon} J_{T, \beta, \varepsilon} \sigma \right\|_0^2 &\geq \left( \frac{\eta}{4\beta^2} - C'_1 \left( \frac{1}{\beta} + \frac{\varepsilon^\delta}{\beta^2} + \frac{\varepsilon}{\beta^4} \right) \right) \int_{\widetilde{s}(M)} |\sigma|^2 dv_{\widetilde{s}(M)} \\ &+ \frac{\varepsilon^\delta}{8\beta^2} \sum_{k=1}^q \int_{\widetilde{s}(M)} \left| Q \nabla_{\widehat{f}_k}^{\widehat{\mathcal{F}}, \phi_1(\widehat{\mathcal{F}}_1^\perp), \beta, \varepsilon} (\tau \sigma) \right|^2 dv_{\widetilde{s}(M)} + \frac{\varepsilon^2}{16} \sum_{k=q+1}^{q+q_1} \int_{\widetilde{s}(M)} \left| Q \nabla_{\widehat{f}_k}^{\widehat{\mathcal{F}}, \phi_1(\widehat{\mathcal{F}}_1^\perp), \beta, \varepsilon} (\tau \sigma) \right|^2 dv_{\widetilde{s}(M)} \\ &- \frac{C_{\beta, \varepsilon}}{\sqrt{T}} \int_{\widetilde{s}(M)} \left( |\sigma|^2 + \sum_{k=1}^{q+q_1} \left| Q \nabla_{\widehat{f}_k}^{\widehat{\mathcal{F}}, \phi_1(\widehat{\mathcal{F}}_1^\perp), \beta, \varepsilon} (\tau \sigma) \right|^2 \right) dv_{\widetilde{s}(M)}, \end{aligned}$$

which holds for all sections  $\sigma \in \Gamma((S(\widehat{\mathcal{F}}) \otimes \Lambda^*(\widehat{\mathcal{F}}_1^\perp) \otimes \phi_1(\widehat{\mathcal{F}}_1^\perp))|_{\widetilde{s}(M)})$ , when  $\beta, \varepsilon > 0$  are small enough.

This refines (2.20) significantly.



**2.10. Proof of Theorem 0.4.** We assume first that  $\dim M$  and  $\text{rk}(\mathcal{F})$  are divisible by 8.

Take  $\varepsilon = \beta^3$  in (2.164). One sees that when  $\beta > 0$  is sufficiently small, one has

$$(2.165) \quad \left\| p_{T,\beta,\varepsilon} D^{\widehat{\mathcal{F}},\phi_1(\widehat{\mathcal{F}}_1^\perp),\beta,\varepsilon} J_{T,\beta,\varepsilon} \sigma \right\|_0^2 \geq \frac{\eta}{8\beta^2} \int_{\widetilde{s}(M)} |\sigma|^2 dv_{\widetilde{s}(M)} \\ + \frac{\beta^{3\delta}}{8\beta^2} \sum_{k=1}^q \int_{\widetilde{s}(M)} \left| Q \nabla_{\widehat{f}_k}^{\widehat{\mathcal{F}},\phi_1(\widehat{\mathcal{F}}_1^\perp),\beta,\varepsilon} (\tau\sigma) \right|^2 dv_{\widetilde{s}(M)} + \frac{\beta^6}{16} \sum_{k=q+1}^{q+q_1} \int_{\widetilde{s}(M)} \left| Q \nabla_{\widehat{f}_k}^{\widehat{\mathcal{F}},\phi_1(\widehat{\mathcal{F}}_1^\perp),\beta,\varepsilon} (\tau\sigma) \right|^2 dv_{\widetilde{s}(M)} \\ - \frac{C_{\beta,\varepsilon}}{\sqrt{T}} \int_{\widetilde{s}(M)} \left( |\sigma|^2 + \sum_{k=1}^{q+q_1} \left| Q \nabla_{\widehat{f}_k}^{\widehat{\mathcal{F}},\phi_1(\widehat{\mathcal{F}}_1^\perp),\beta,\varepsilon} (\tau\sigma) \right|^2 \right) dv_{\widetilde{s}(M)}.$$

By fixing  $\beta > 0$  and taking  $T > 0$  large enough, one sees that there exist  $\beta_0 > 0$ ,  $T_0 > 0$  such that the following inequality holds,

$$(2.166) \quad \left\| p_{T,\beta,\varepsilon} D^{\widehat{\mathcal{F}},\phi_1(\widehat{\mathcal{F}}_1^\perp),\beta,\varepsilon} J_{T_0,\beta,\varepsilon} \sigma \right\|_0^2 \geq \frac{\eta}{16\beta_0^2} \int_{\widetilde{s}(M)} |\sigma|^2 dv_{\widetilde{s}(M)} \\ + \frac{\beta_0^{3\delta}}{16\beta_0^2} \sum_{k=1}^q \int_{\widetilde{s}(M)} \left| Q \nabla_{\widehat{f}_k}^{\widehat{\mathcal{F}},\phi_1(\widehat{\mathcal{F}}_1^\perp),\beta,\varepsilon} (\tau\sigma) \right|^2 dv_{\widetilde{s}(M)} + \frac{\beta_0^6}{32} \sum_{k=q+1}^{q+q_1} \int_{\widetilde{s}(M)} \left| Q \nabla_{\widehat{f}_k}^{\widehat{\mathcal{F}},\phi_1(\widehat{\mathcal{F}}_1^\perp),\beta,\varepsilon} (\tau\sigma) \right|^2 dv_{\widetilde{s}(M)}.$$

Let  $\beta_0 > 0$ ,  $T_0 > 0$  be fixed as in (2.166). Let

$$(2.167) \quad D_{\widetilde{s}(M),\beta_0,T_0} : \Gamma \left( \left( S(\widehat{\mathcal{F}}) \widehat{\otimes} \Lambda^* (\widehat{\mathcal{F}}_1^\perp) \otimes \phi_1 (\widehat{\mathcal{F}}_1^\perp) \right) \Big|_{\widetilde{s}(M)} \right) \\ \longrightarrow \Gamma \left( \left( S(\widehat{\mathcal{F}}) \widehat{\otimes} \Lambda^* (\widehat{\mathcal{F}}_1^\perp) \otimes \phi_1 (\widehat{\mathcal{F}}_1^\perp) \right) \Big|_{\widetilde{s}(M)} \right)$$

be the operator defined by

$$(2.168) \quad D_{\widetilde{s}(M),\beta_0,T_0} = J_{T_0,\beta_0}^{-1} p_{T_0,\beta_0} D^{\widehat{\mathcal{F}},\phi_1(\widehat{\mathcal{F}}_1^\perp),\beta_0} p_{T_0,\beta_0} J_{T_0,\beta_0},$$

where as we take now  $\varepsilon = \beta^3$ , we omit the subscript  $\varepsilon$ .

Since  $J_{T_0,\beta_0}$  is an isometry, by (2.166) and (2.168),  $D_{\widetilde{s}(M),\beta_0,T_0}$  is a formally self-adjoint elliptic operator. Moreover,

$$(2.169) \quad \ker (D_{\widetilde{s}(M),\beta_0,T_0}) = 0.$$

Let

$$(2.170) \quad D_{\widetilde{s}(M),\beta_0,T_0,+} : \Gamma \left( \left( \left( S(\widehat{\mathcal{F}}) \widehat{\otimes} \Lambda^* (\widehat{\mathcal{F}}_1^\perp) \right)_+ \otimes \phi_1 (\widehat{\mathcal{F}}_1^\perp) \right) \Big|_{\widetilde{s}(M)} \right) \\ \longrightarrow \Gamma \left( \left( \left( S(\widehat{\mathcal{F}}) \widehat{\otimes} \Lambda^* (\widehat{\mathcal{F}}_1^\perp) \right)_- \otimes \phi_1 (\widehat{\mathcal{F}}_1^\perp) \right) \Big|_{\widetilde{s}(M)} \right)$$

be the obvious restriction, then by (2.169), one has

$$(2.171) \quad \text{ind} (D_{\widetilde{s}(M),\beta_0,T_0,+}) = 0.$$

Since  $\widehat{\mathcal{F}}|_{\widetilde{s}(M)} \simeq \widetilde{s}_* F \subset T\widetilde{s}(M)$  and  $\widehat{\mathcal{F}}_1^\perp|_{\widetilde{s}(M)} \simeq \widetilde{s}_*(TM/F) \simeq T\widetilde{s}(M)/\widetilde{s}_* F$ , by (2.168) and (2.170), one sees that  $D_{\widetilde{s}(M),\beta_0,T_0,+}$  is homotopic to the corresponding sub-Dirac

operator on  $\tilde{s}(M)$  (and thus on  $(M, F)$ ) constructed in [18, Definition 2.2]. Thus, they have the same index. In particular, by the Atiyah-Singer index theorem [2], one gets (compare with (1.74) and [18, (2.44)])

$$(2.172) \quad \text{ind} \left( D_{\tilde{s}(M), \beta_0, T_0, +} \right) = 2^{\frac{q_1}{2}} \left\langle \hat{A}(F) \hat{L}(TM/F) \text{ch}(\phi_1(TM/F)), [M] \right\rangle.$$

By (2.171) and (2.172), one gets

$$(2.173) \quad \left\langle \hat{A}(F) \hat{L}(TM/F) \text{ch}(\phi_1(TM/F)), [M] \right\rangle = 0,$$

which implies that for any Pontrjagin class  $p(TM/F)$  of  $TM/F$ , one has

$$(2.174) \quad \left\langle \hat{A}(F) p(TM/F), [M] \right\rangle = 0.$$

By taking  $p(TM/F) = \hat{A}(TM/F)$ , one gets the vanishing of  $\hat{A}(M)$ .

Now if one of  $\dim M$  and  $\text{rk}(F)$  is not divisible by 8, then we simply work on  $M \times \cdots \times M$  (8 times) to get the result.

The proof of Theorem 0.4 is completed.

**2.11. Proof of Theorem 0.1: the case of dimension  $4k$ .** Without loss of generality we assume that  $\dim M$ ,  $\text{rk}(F)$  are divisible by 8. We assume first that  $F$  and thus  $F^\perp \simeq TM/F$  also are oriented.

Since  $M$  is spin,  $\hat{\mathcal{F}} \oplus \hat{\mathcal{F}}_1^\perp = \tilde{\pi}^*(TM)$  is spin over  $\widehat{\mathcal{M}}$ . Thus, we consider directly the sub-Dirac operator (and its deformations) acting on smooth sections of  $S(\hat{\mathcal{F}} \oplus \hat{\mathcal{F}}_1^\perp) \hat{\otimes} \Lambda^*(\hat{\mathcal{F}}_2^\perp) \otimes \phi(\hat{\mathcal{F}}_1^\perp)$ , where  $S(\hat{\mathcal{F}} \oplus \hat{\mathcal{F}}_1^\perp)$  is the corresponding bundle of spinors. Then everything in the previous subsections still works, and we get the vanishing result

$$(2.175) \quad \left\langle \hat{A}(TM) \text{ch}(\phi(TM/F)), [M] \right\rangle = 0,$$

from which one deduces that

$$(2.176) \quad \left\langle \hat{A}(F) p(TM/F), [M] \right\rangle = 0$$

for any Pontrjagin class  $p(TM/F)$  of  $TM/F$ . In particular, Theorem 0.1 holds.

Now if  $F$  is not orientable, then one can consider the double covering of  $M$  with respect to  $w_1(F)$ , the first Stiefel-Whitney class of  $F$ . Then one applies Theorem 0.1 to this double covering and get the same result on  $M$  by the multiplicativity of the  $\hat{A}$ -genus.

**2.12. Vanishing of the mod 2 index.** In this subsection, we will prove Theorem 0.1 for the case of  $\dim M = 8k + 1$ . The proof for the case of  $\dim M = 8k + 2$  is similar.

So from now on we assume that  $\dim M = 8k + 1$  ( $k \geq 1$ ). Then  $\dim \widehat{\mathcal{M}} = \dim M + q_1(q_1 + 1)$ , where  $q_1 = \dim M - \text{rk}(F)$  is the codimension of  $F$ . We assume first that  $F$  and thus  $F^\perp$  are oriented. Set  $\widetilde{\mathcal{M}} = \widehat{\mathcal{M}} \times \mathbf{R}^{7q_1(q_1+1)}$ , then one has

$$(2.177) \quad \dim \widetilde{\mathcal{M}} \equiv 1 \pmod{8}.$$

Recall that in this dimension one can construct *real* spinor representations (cf. [14]).

We lift everything to  $\widetilde{\mathcal{M}}$  and use “ $\sim$ ” to decorate the obvious modifications.

We assume that  $TM$ ,  $F$  and  $F^\perp \simeq TM/F$  are oriented and that  $M$  is spin and carries a fixed spin structure. Then  $\widetilde{\mathcal{F}} \oplus \widetilde{\mathcal{F}}_1^\perp$  carries a canonical spin structure.

As in Section 2.11, we consider the sub-Dirac operator

$$(2.178) \quad D^{\widetilde{\mathcal{M}}} : \Gamma \left( S \left( \widetilde{\mathcal{F}} \oplus \widetilde{\mathcal{F}}_1^\perp \right) \widehat{\otimes} \Lambda^* \left( \widetilde{\mathcal{F}}_2^\perp \right) \right) \longrightarrow \Gamma \left( S \left( \widetilde{\mathcal{F}} \oplus \widetilde{\mathcal{F}}_1^\perp \right) \widehat{\otimes} \Lambda^* \left( \widetilde{\mathcal{F}}_2^\perp \right) \right).$$

For any  $\beta > 0$ , let  $D^{\widetilde{\mathcal{M}},\beta}$  denote the sub-Dirac operator in (2.178) with respect to the deformed metric (2.5).<sup>13</sup>

Let  $e_1, \dots, e_{\widetilde{q}_2}$  be an oriented orthonormal basis of  $\widetilde{\mathcal{F}}_2^\perp$ . Let  $e^1, \dots, e^{\widetilde{q}_2}$  be the dual basis. Recall that here  $\widetilde{q}_2 = 8q_1(q_1 + 1)$ .

Let  $L$  be the trivial real line bundle generated by the element  $1 + e^1 \wedge \dots \wedge e^{\widetilde{q}_2} \in \Lambda^*(\widetilde{\mathcal{F}}_2^\perp)$ . We may also view  $L$  as a sub-line bundle of  $\Lambda^*(\widetilde{\mathcal{F}}_2^\perp)$ .

Let  $Q_L : \Lambda^*(\widetilde{\mathcal{F}}_2^\perp) \rightarrow L$  denote the orthogonal projection from  $\Lambda^*(\widetilde{\mathcal{F}}_2^\perp)$  to  $L$ .

Let  $s' : M \hookrightarrow \widetilde{\mathcal{M}} = \widehat{\mathcal{M}} \times \mathbf{R}^{7q_1(q_1+1)}$  be the embedding defined by  $s'(x) = \widetilde{s}(x) \times \{0\}$ .

For any  $T > 0$ ,  $0 < \beta \leq 1$ , let  $J_{T,\beta}$  be defined as in (2.17), with respect to the embedding  $s'$ .

Let

$$(2.179) \quad J_{T,\beta}^L : \Gamma \left( \left( S \left( \widetilde{\mathcal{F}} \oplus \widetilde{\mathcal{F}}_1^\perp \right) \right) \Big|_{s'(M)} \right) \longrightarrow \Gamma \left( S \left( \widetilde{\mathcal{F}} \oplus \widetilde{\mathcal{F}}_1^\perp \right) \widehat{\otimes} \Lambda^* \left( \widetilde{\mathcal{F}}_2^\perp \right) \right)$$

be defined by

$$(2.180) \quad J_{T,\beta}^L : \sigma \mapsto (J_{T,\beta} \sigma) \frac{1 + e^1 \wedge \dots \wedge e^{\widetilde{q}_2}}{\sqrt{2}}.$$

Then  $J_{T,\beta}^L$  is still an isometric embedding. Let  $p_{T,\beta}^L$  be the orthogonal projection from the  $L^2$ -completion of  $\Gamma(S(\widetilde{\mathcal{F}} \oplus \widetilde{\mathcal{F}}_1^\perp) \widehat{\otimes} \Lambda^*(\widetilde{\mathcal{F}}_2^\perp))$  to the  $L^2$ -completion of  $\text{Im}(J_{T,\beta}^L)$ .

One verifies directly that

$$(2.181) \quad Q_L \left( c_\beta(e_1) \cdots c_\beta(e_{\widetilde{q}_2}) (1 + e^1 \wedge \dots \wedge e^{\widetilde{q}_2}) \right) = 1 + e^1 \wedge \dots \wedge e^{\widetilde{q}_2}.$$

Let  $h_1, \dots, h_{\dim \widetilde{\mathcal{M}}}$  be an oriented orthonormal basis of  $T\widetilde{\mathcal{M}}$ . Set

$$(2.182) \quad \widehat{\tau} = c(h_1) \cdots c(h_{\dim \widetilde{\mathcal{M}}}).$$

Let  $\widetilde{D}_T^{\widetilde{\mathcal{M}},\beta} : \Gamma((S(\widetilde{\mathcal{F}} \oplus \widetilde{\mathcal{F}}_1^\perp))|_{s'(M)}) \rightarrow \Gamma((S(\widetilde{\mathcal{F}} \oplus \widetilde{\mathcal{F}}_1^\perp))|_{s'(M)})$  be defined by

$$(2.183) \quad \widetilde{D}_T^{\widetilde{\mathcal{M}},\beta} = (J_{T,\beta}^L)^{-1} p_{T,\beta}^L \widehat{\tau} \widetilde{D}^{\widetilde{\mathcal{M}},\beta} p_{T,\beta}^L J_{T,\beta}^L.$$

Since  $\dim M = \text{rk}(\widetilde{\mathcal{F}} \oplus \widetilde{\mathcal{F}}_1^\perp) \equiv 1 \pmod{8}$ , one can consider real spinor bundle  $S(\widetilde{\mathcal{F}} \oplus \widetilde{\mathcal{F}}_1^\perp)$  as well as the real exterior algebra  $\Lambda^*(\widetilde{\mathcal{F}}_2^\perp)$ . Thus one can view  $\widetilde{D}_T^{\widetilde{\mathcal{M}},\beta}$  as a real operator acting on  $\Gamma((S(\widetilde{\mathcal{F}} \oplus \widetilde{\mathcal{F}}_1^\perp))|_{s'(M)})$ . Moreover, by (2.6), (2.177), (2.182) and (2.183), one verifies that  $\widetilde{D}_T^{\widetilde{\mathcal{M}},\beta}$  is a real formally skew-adjoint elliptic operator which is homotopic to the corresponding real skew-adjoint Dirac operator on  $s'(M)$  defined in [3]. Thus  $\dim(\ker \widetilde{D}_T^{\widetilde{\mathcal{M}},\beta}) \pmod{2}$  is a smooth invariant which, by the homotopy invariance of the mod 2 index in the sense of Atiyah-Singer, can be identified with the Atiyah-Milnor-Singer  $\alpha$ -invariant (cf. [3, Section 3]).

<sup>13</sup>Recall that we now take  $\varepsilon = \beta^3$ .

In summary, we have

$$(2.184) \quad \alpha(M) = \text{ind}_2 \left( \widetilde{D}_T^{\mathcal{M}, \beta} \right) \equiv \dim \left( \ker \widetilde{D}_T^{\mathcal{M}, \beta} \right) \pmod{2}.$$

Now by using (2.180), (2.181) and by proceeding as in the previous subsections, one can show that there exist  $0 < \beta_1 \leq 1$  and  $T_1 \geq 1$  such that for any  $T \geq T_1$ , one has

$$(2.185) \quad \dim \left( \ker \widetilde{D}_T^{\mathcal{M}, \beta_1} \right) = 0.$$

Combining with (2.184), we get

$$(2.186) \quad \alpha(M) = 0.$$

Thus Theorem 0.1 holds.

If  $F$  and thus  $F^\perp$  are not orientable, then we take  $\widetilde{\mathcal{M}}$  to be the direct sum of the orientation line bundle  $o(\widehat{\mathcal{F}}_2^\perp)$  and the trivial vector bundle  $\mathbf{R}^{7q_1(q_1+1)-1}$  over  $\widehat{\mathcal{M}}$ . Then one sees that  $\widetilde{\mathcal{F}}_2^\perp$  is still orientable, and one can then proceed as above to complete the proof of Theorem 0.1.

**2.13. Proof of Theorem 0.5.** From our positivity result (2.166), which can well be used to replace the Lichnerowicz positivity used in [9, Proof of Theorem 2.1], one can proceed as in [9] to get the same conclusion of [9, Theorem 2.1] and [9, Corollary 2.2] under the condition of Theorem 0.5. In particular, Theorem 0.5 holds.

We leave the details and other immediate generalizations to interested readers.

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